QUALIFYING EXAM IN ALGEBRA

August 2003

1. There are 18 problems on the exam. Work and turn in 10 problems, in the following categories.

I.	Linear Algebra	—	1 problem
II.	Group Theory	—	3 problems
III.	Ring Theory	—	2 problems
IV.	Field Theory	—	3 problems
Any of the four areas		—	1 problem

2. Turn in only 10 problems. No credit will be given for extra problems. All problems are weighted equally.

3. Put each problem on a separate sheet of paper, and write only on one side. Put your name on each page.

4. If you feel there is a misprint or error in the statement of a problem, then interpret it in such a way that the problem is not trivial.

I. Linear Algebra

1. Find all possible Jordan canonical forms of a complex matrix with characteristic polynomial $(x-4)^3(x-2)^2$.

2. (Fitting's Lemma for vector spaces) Let $\varphi : V \to V$ be a linear transformation of a finite dimensional vector space to itself. Prove that there exists a decomposition of V as $V = U \oplus W$, where each summand is φ-invariant, $\varphi|_U$ is nilpotent, and $\varphi|_W$ is nonsingular.

3. Let V be a vector space and let U and W be finite dimensional subspaces of V. Show that $\dim(U+W) = \dim U + \dim W - \dim U \cap W$.

II. Group Theory

1. (a) Give an example of two nonconjugate elements of S_7 that have the same order.

   (b) If $g \in S_7$ has maximal order, what is $o(g)$?

   (c) Does the element g that you found in part (b) lie in A_7?

   (d) Is the set $\{h \in S_7 | o(h) = o(g)\}$ a single conjugacy class in S_7, where g is the elment found in part (b)?

2. Let G be a finite p-group for some prime p. Show that if G is not cyclic, then G has at least $p+1$ maximal subgroups.

3. Prove that a group G of order 36 must have a normal subgroup of order 3 or 9.

4. Let the group G act transitively on the set Ω, and let N be a normal subgroup of G. Prove that G permutes the N-orbits of Ω and that these orbits all have the same size.

5. Suppose a group G has a subgroup H with $|G : H| = n < \infty$. Prove that G has a normal subgroup N with $N \subseteq H$ and $|G : N| \leq n!$.

III. Ring Theory

1. Let R be the ring of all 2×2 matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ where a and b are real numbers. Prove that R is isomorphic to $\boldsymbol{C}$, the field of complex numbers.

2. Let R be a principal ideal domain. Show that if (a) is a nonzero ideal in R, then there are only finitely many ideals in R containing (a).

3. Let $F[x, y]$ be the polynomial ring over a field F in two indeterminates x, y. Show that the ideal generated by $\{x, y\}$ is not a principal ideal.

4. Let R be a commutative ring with identity that has exactly three ideals, $\{0\}$, I, and R.
   (a) Show that if $a \notin I$, then a is a unit of R.
   (b) Show that if $a, b \in I$, then $ab = 0$.

5. Let D be an integral domain and F its field of fractions. Let P be a prime ideal in D and $D_P = \{\, ab^{-1} \mid a, b \in D,\ b \notin P \,\} \subseteq F$. Show that D_P has a unique maximal ideal.

IV. Field Theory

1. Find the minimal polynomial of $\alpha = \sqrt{3 + 2\sqrt{2}}$ over the field $\boldsymbol{Q}$ of rational numbers, and *prove* it is the minimal polynomial.

2. Let F be a field, let $E = F(a)$ be a simple extension field of F, and let $b \in E - F$. Prove that a is algebraic over $F(b)$.

3. Let F be a field and E a splitting field of the irreducible polynomial $f(x) \in F[x]$. Show that if $c, d \in F$ and $c \neq 0$, then the polynomial $f(cx + d)$ splits in $E[x]$.

4. Let G be a finite group. Show that there is a field F and a Galois extension K of F such that $G \cong \mathrm{Gal}(K/F)$.

5. Let E and F be finite subfields of a field K. Show that if E and F have the same number of elements, then $E = F$.