QUALIFYING EXAM IN ALGEBRA

August 2004

1. There are 18 problems on the exam. Work and turn in 10 problems, in the following categories.

I.	Linear Algebra	—	1 problem
II.	Group Theory	—	3 problems
III.	Ring Theory	—	2 problems
IV.	Field Theory	—	3 problems
Any of the four areas		—	1 problem

2. Turn in only 10 problems. No credit will be given for extra problems. All problems are weighted equally.

3. Put each problem on a separate sheet of paper, and write only on one side. Put your name on each page.

4. If you feel there is a misprint or error in the statement of a problem, then interpret it in such a way that the problem is not trivial.

I. Linear Algebra

1. Let $A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$.

   (a) Verify that the characteristic polynomial of A is $\Delta(x) = x(x-1)^2$.

   (b) For each eigenvalue λ of A, find a basis for the eigenspace E_λ.

   (c) Determine if A is diagonalizable. If so, give matrices P, B such that $P^{-1}AP = B$ and B is diagonal. If not, explain carefully *why* A is not diagonalizable.

2. Let A be a strictly upper triangular $n \times n$ matrix with real entries, and let I be the $n \times n$ identity matrix. Show that $I - A$ is invertible and express the inverse of $I - A$ as a function of A.

3. Let V be the vector space of $n \times n$ matrices over the field $\mathbb{R}$ of real numbers. Let U be the subspace of V consisting of symmetric matrices and W the subspace of V consisting of skew-symmetric matrices. Show that $V = U \oplus W$.

II. Group Theory

1. Let N be a normal subgroup of G and let $\mathcal{C}$ be a conjugacy class of G that is contained in N. Prove that if $|G : N| = p$ is prime, then either $\mathcal{C}$ is a conjugacy class of N or $\mathcal{C}$ is a union of p distinct conjugacy classes of N.

2. Show that if G is a subgroup of S_n of index 2, then $G = A_n$.

3. Let G be a simple group of order greater than 2 and let $\mathrm{Aut}(G)$ be its automorphism group. Show that the center of $\mathrm{Aut}(G)$ is trivial if and only if G is non-abelian.

4. Show that a group of order $2004 = 2^2 \cdot 3 \cdot 167$ must be solvable. Give an example of a group of order 2004 in which a Sylow 3-subgroup is not a normal subgroup.

5. Show that if P is a finite p-group and $\langle 1 \rangle \neq N \trianglelefteq P$, then $N \cap Z(P) \neq \langle 1 \rangle$.

III. Ring Theory

1. (a) Let R be a commutative ring with 1. Show that if M is a maximal ideal of R then M is a prime ideal of R.

   (b) Give an example of a non-zero prime ideal in a ring R that is not a maximal ideal.

2. (a) Let $\mathbb{Z}\left[\frac{1}{2}\right] = \left\{ \frac{a}{2^n} \mid a, n \in \mathbb{Z},\, n \geqslant 0 \right\}$, the smallest subring of $\mathbb{Q}$ containing $\mathbb{Z}$ and $\frac{1}{2}$. Let $(2x-1)$ be the ideal of $\mathbb{Z}[x]$ generated by the polynomial $2x-1$. Show that $\mathbb{Z}[x]/(2x-1) \cong \mathbb{Z}\left[\frac{1}{2}\right]$.

   (b) Find an ideal I of $\mathbb{Z}[x]$ such that $(2x-1) \subsetneq I \subsetneq \mathbb{Z}[x]$.

3. Let R be a commutative ring with 1 and let I and J be ideals of R such that $I+J=R$. Show that $R/(I \cap J) \cong R/I \oplus R/J$.

4. Show that if p is a prime such that there is an integer b with $p = b^2 + 4$, then $\mathbb{Z}[\sqrt{p}]$ is not a unique factorization domain.

5. Let R be a commutative ring with 1. We say R satisfies the *ascending chain condition* if whenever $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ is an ascending chain of ideals, there is an integer N such that $I_k = I_N$ for all $k \geqslant N$. Show that R satisfies the ascending chain condition if and only if every ideal of R is finitely generated.

IV. Field Theory

1. Suppose K is an algebraic extension field of a field F such that there are only finitely many intermediate fields between F and K. Show that K is a simple extension of F.

2. Let F be a field, $f(x)$ an irreducible polynomial in $F[x]$, and α a root of f in some extension of F. Show that if some odd degree term of $f(x)$ has a non-zero coefficient, then $F(\alpha) = F(\alpha^2)$.

3. Let p be a prime such that there is a positive integer d with $p = 1 + d^2$ and let $\alpha = \sqrt{p + d\sqrt{p}}$. Show that $\mathbb{Q}(\alpha)$ is a cyclic Galois extension of $\mathbb{Q}$ of degree 4. Find all fields F with $\mathbb{Q} \subseteq F \subseteq \mathbb{Q}(\alpha)$.
   [Hint: Show that $f(x) = x^4 - 2px^2 + p$ is the minimal polynomial of α over $\mathbb{Q}$ and that the roots of f are $\pm\alpha$, $\pm\frac{\sqrt{p}}{\alpha}$.]

4. Let ϵ be a primitive n-th root of unity over $\mathbb{Q}$, where $n > 2$, and let $\alpha = \epsilon + \epsilon^{-1}$. Prove that α is algebraic over $\mathbb{Q}$ of degree $\varphi(n)/2$.

5. Let α be a root of x^2+1 in an extension of $\mathbb{Z}_3$, $K = \mathbb{Z}_3(\alpha)$, and let $f(x) = x^4+1 \in \mathbb{Z}_3[x]$.
   (a) Show that f splits over K.
   (b) Find a generator β of the multiplicative group K^* of K.
   (c) Express the roots of f in terms of β.