

QUALIFYING EXAM IN ALGEBRA

August 2006

1. There are 18 problems on the exam. Work and turn in 10 problems, in the following categories.
 - I. Linear Algebra — 1 problem
 - II. Group Theory — 3 problems
 - III. Ring Theory — 2 problems
 - IV. Field Theory — 3 problems
 - Any of the four areas — 1 problem
2. Turn in only 10 problems. No credit will be given for extra problems. All problems are weighted equally.
3. Put each problem on a separate sheet of paper, and write only on one side. Put your name on each page.
4. If you feel there is a misprint or error in the statement of a problem, then interpret it in such a way that the problem is not trivial.

I. Linear Algebra

1. Let $A = \begin{bmatrix} 5 & 0 & 6 \\ 2 & 2 & 4 \\ -2 & 0 & -2 \end{bmatrix}$.

- (a) Verify that the characteristic polynomial of A is $\Delta(x) = (x - 1)(x - 2)^2$.
 - (b) For each eigenvalue λ of A , find a basis for the eigenspace E_λ .
 - (c) Determine if A is diagonalizable. If so, give matrices P, B such that $P^{-1}AP = B$ and B is diagonal. If not, explain carefully *why* A is not diagonalizable.
2. Let V be a finite dimensional vector space and $T : V \rightarrow V$ a *non-zero* linear operator. Show that if $\ker T = \text{Im } T$, then $\dim V$ is an *even* integer and the minimal polynomial of T is $m(x) = x^2$.
3. Let A and B be complex $n \times n$ matrices. Prove that if $AB = BA$, then A and B share a common eigenvector.

II. Group Theory

1. Let G be a group with a normal subgroup N of order 7, such that $G/N \cong D_{10}$, the dihedral group of order 10. Show that $|G| = 70$, G has a normal subgroup of order 35, and G has 5 subgroups of order 14 that are not normal.
2. Let τ be an element of the symmetric group S_n and let $\sigma \in S_n$ be a transposition. Show that the number of cycles in the cycle decomposition of $\sigma\tau$ is either one more or one less than the number of cycles in the cycle decomposition of τ .
3. Let G be a group acting on a set Ω . Show that the following are equivalent.
 - (i) The action is doubly transitive (i.e., for any two ordered pairs $(\alpha_1, \beta_1), (\alpha_2, \beta_2)$ of elements of Ω with $\alpha_1 \neq \beta_1$ and $\alpha_2 \neq \beta_2$, there is an element g in G such that $g \cdot \alpha_1 = \alpha_2$ and $g \cdot \beta_1 = \beta_2$).
 - (ii) For all $\alpha \in \Omega$, the stabilizer G_α acts transitively on $\Omega - \{\alpha\}$.
4. Let P be a finite p -group and let H be a proper subgroup of P . Prove that H is a proper subgroup of its normalizer $N_P(H)$.
5. Show that if G is a group of order $2010 = 2 \cdot 3 \cdot 5 \cdot 67$, then G has a normal subgroup of order 5.

III. Ring Theory

1. Show that a non-zero ring R in which $x^2 = x$ for all $x \in R$ is of characteristic 2 and is commutative.
2. Let D be a principal ideal domain.
 - (a) For $a, b \in D$, define a *least common multiple* of a and b .
 - (b) Show that $d \in D$ is a least common multiple of a and b if and only if $(a) \cap (b) = (d)$.
3. Let $D = \mathbb{Z}(\sqrt{-11}) = \{m + n\sqrt{-11} \mid m, n \in \mathbb{Z}\}$ and $F = \mathbb{Q}(\sqrt{-11})$ its field of fractions. Show the following:
 - (a) $x^2 - x + 3$ is irreducible in $D[x]$ but not in $F[x]$.
 - (b) D is not a unique factorization domain.
4. Let R be a commutative ring with $1 \neq 0$ in which the set of nonunits is closed under addition. Prove that R is local, i.e., has a unique maximal ideal.
5. Let R be a commutative ring with identity. An ideal I of R is *irreducible* if it cannot be expressed as the intersection of two ideals of R neither of which is contained in the other. Show the following.
 - (a) If P is a prime ideal then P is irreducible.
 - (b) If x is a non-zero element of R , then there is an ideal I_x , maximal with respect to the property that $x \notin I_x$, and I_x is irreducible.

IV. Field Theory

1. Let K be an extension field of F with $a, b \in K$. Let $[F(a) : F] = m$ and $[F(b) : F] = n$ and assume $(m, n) = 1$. Show that $F(a) \cap F(b) = F$ and $[F(a, b) : F] = mn$.
2. Show that if K is algebraic over F and $\sigma : K \rightarrow K$ is an F -monomorphism, then σ is onto.
3. Let K be a finite normal extension of F and let E be the fixed field of the group of all F -automorphisms of K . Show that the minimal polynomial over F of each element of E has only one distinct root.
4. Let $\mathbb{F} = \mathbb{F}_{81}$ be the field of 81 elements.
 - (a) Find all subfields of \mathbb{F} .
 - (b) Determine the number of primitive elements for \mathbb{F} over the field \mathbb{F}_3 of 3 elements (i.e., elements α of \mathbb{F} such that $\mathbb{F} = \mathbb{F}_3(\alpha)$).
 - (c) Find the number of generators for the multiplicative group \mathbb{F}^* of \mathbb{F} (i.e., elements β of \mathbb{F} such that $\langle \beta \rangle = \mathbb{F}^*$).
5. Let $\alpha = \sqrt{2} + \sqrt{3}$ and let $E = \mathbb{Q}(\alpha)$.
 - (a) Find the minimal polynomial $m(x)$ of α over \mathbb{Q} and $|E : \mathbb{Q}|$.
 - (b) Find the splitting field of $m(x)$ over \mathbb{Q} and all intermediate fields, and find its Galois group over \mathbb{Q} and all its subgroups.