QUALIFYING EXAM IN ALGEBRA
August 2007

1. There are 18 problems on the exam. Work and turn in 10 problems, in the following categories.

   I. Linear Algebra — 1 problem
   II. Group Theory — 3 problems
   III. Ring Theory — 2 problems
   IV. Field Theory — 3 problems
   Any of the four areas — 1 problem

2. Turn in only 10 problems. No credit will be given for extra problems. All problems are weighted equally.

3. Put each problem on a separate sheet of paper, and write only on one side. Put your name on each page.

4. If you feel there is a misprint or error in the statement of a problem, then interpret it in such a way that the problem is not trivial.
I. Linear Algebra

1. A square matrix $A$ over $\mathbb{C}$ is Hermitian if $\bar{A}^t = A$. Prove that the eigenvalues of a Hermitian matrix are all real.

2. A matrix $A$ has characteristic polynomial $\Delta(x) = (x - 3)^5$ and minimal polynomial $m(x) = (x - 3)^3$.
   (a) List all possible Jordan canonical forms for $A$.
   (b) Determine the Jordan canonical form of the matrix
   \[
   A = \begin{bmatrix}
   3 & -1 & 2 & 0 & 0 \\
   2 & 3 & 0 & -2 & 0 \\
   1 & 0 & 3 & -1 & 0 \\
   0 & -1 & 2 & 3 & 0 \\
   0 & 2 & -3 & 0 & 3
   \end{bmatrix},
   \]
   which has the given characteristic and minimal polynomials.

3. Let $V$ be the vector space over the field $\mathbb{R}$ of real numbers consisting of all functions from $\mathbb{R}$ into $\mathbb{R}$. Let $U$ be the subspace of even functions and $W$ the subspace of odd functions. Show that $V = U \oplus W$. 
II. Group Theory

1. Let $G$ be a group and let $Z(G)$ be the center of $G$. Prove the following.
   
   (a) If $G/Z(G)$ is cyclic, then $G$ is abelian.
   
   (b) If $G$ is of order $p^2$, where $p$ is a prime, then $G$ is abelian.


3. Let $\sigma = (1\ 2\ 3)(4\ 5\ 6) \in S_6$.
   
   (a) Determine the size of the conjugacy class of $\sigma$ and the order of the centralizer of $\sigma$ in $S_6$.
   
   (b) Determine if $C_{S_6}(\sigma)$ is abelian or non-abelian. Prove your answer.

4. Let $G$ be a finite group. Show that if $G$ has a normal subgroup $N$ of order 3 that is not contained in the center of $G$, then $G$ has a subgroup of index 2. [Hint: The group $G$ acts on $N$ by conjugation.]

5. Show that a group of order 96 must have a normal subgroup of order 16 or 32.
III. Ring Theory

1. Let $R$ be the ring of all $2 \times 2$ matrices of the form \[
\begin{bmatrix}
a & b \\
-b & a
\end{bmatrix}
\] where $a$ and $b$ are real numbers. Prove that $R$ is isomorphic to $\mathbb{C}$, the field of complex numbers.

2. Let $R$ be a commutative ring with identity. Show that if $x$ and $y$ are nilpotent elements of $R$ then $x + y$ is nilpotent and the set of all nilpotent elements is an ideal in $R$.

3. Show that the ring $\mathbb{Z}[i]$ of Gaussian integers is a Euclidean ring and compute the greatest common divisor of $5 + i$ and $13$ using the Euclidean algorithm.

4. Show that if $R$ is a commutative ring with $1$ that is not a field, then $R[x]$ is not a principal ideal ring.

5. Let $R$ be an integral domain, $S$ a multiplicative set, and let $S^{-1}R = \{ \frac{r}{s} \mid r \in R, s \in S \}$ (contained in the field of fractions of $R$). Show that if $P$ is a prime ideal of $R$, then $S^{-1}P$ is either a prime ideal of $S^{-1}R$ or else equals $S^{-1}R$.

IV. Field Theory

1. Find the minimal polynomial of $\alpha = \sqrt{3} + \sqrt[4]{7}$ over the field $\mathbb{Q}$ of rational numbers, and prove it is the minimal polynomial.

2. Show that if $K$ is algebraic over $F$ and $\sigma : K \to K$ is an $F$-monomorphism, then $\sigma$ is onto.

3. Prove that the multiplicative group of a finite field must be cyclic.

4. Let $F$ be any field and let $f(x) = x^n - 1 \in F[x]$. Show that if $K$ is the splitting field of $f(x)$ over $F$, then $K$ is separable over $F$ (hence Galois) and that $\text{Gal}(K/F)$ is abelian.

5. (a) Find the Galois group of $x^3 - 2$ over $\mathbb{Q}$ and demonstrate the Galois correspondence between the subgroups of the Galois group and the subfields of the splitting field.

   (b) Find all automorphisms of $\mathbb{Q}(\sqrt[3]{2})$. Is there an $f(x) \in \mathbb{Q}[x]$ with splitting field $\mathbb{Q}(\sqrt[3]{2})$? Explain.