QUALIFYING EXAM IN ALGEBRA

August 2007

1. There are 18 problems on the exam. Work and turn in 10 problems, in the following categories.

I.	Linear Algebra	—	1 problem
II.	Group Theory	—	3 problems
III.	Ring Theory	—	2 problems
IV.	Field Theory	—	3 problems
Any of the four areas		—	1 problem

2. Turn in only 10 problems. No credit will be given for extra problems. All problems are weighted equally.

3. Put each problem on a separate sheet of paper, and write only on one side. Put your name on each page.

4. If you feel there is a misprint or error in the statement of a problem, then interpret it in such a way that the problem is not trivial.

I. Linear Algebra

1. A square matrix A over $\mathbb{C}$ is *Hermitian* if $\bar{A}^t = A$. Prove that the eigenvalues of a Hermitian matrix are all real.

2. A matrix A has characteristic polynomial $\Delta(x) = (x-3)^5$ and minimal polynomial $m(x) = (x-3)^3$.
   (a) List all possible Jordan canonical forms for A.
   (b) Determine the Jordan canonical form of the matrix
   $$A = \begin{bmatrix} 3 & -1 & 2 & 0 & 0 \\ 2 & 3 & 0 & -2 & 0 \\ 1 & 0 & 3 & -1 & 0 \\ 0 & -1 & 2 & 3 & 0 \\ 0 & 2 & -3 & 0 & 3 \end{bmatrix},$$
   which has the given characteristic and minimal polynomials.

3. Let V be the vector space over the field $\mathbb{R}$ of real numbers consisting of all functions from $\mathbb{R}$ into $\mathbb{R}$. Let U be the subspace of even functions and W the subspace of odd functions. Show that $V = U \oplus W$.

II. Group Theory

1. Let G be a group and let $Z(G)$ be the center of G. Prove the following.
   (a) If $G/Z(G)$ is cyclic, then G is abelian.
   (b) If G is of order p^2, where p is a prime, then G is abelian.

2. Let H and K be subgroups of a finite group G satisfying $(|G:H|, |G:K|) = 1$. Show that $|G : H \cap K| = |G : H| \cdot |G : K|$ and $G = HK$.

3. Let $\sigma = (\ 1 \quad 2 \quad 3\ )(\ 4 \quad 5 \quad 6\ ) \in S_6$.
   (a) Determine the size of the conjugacy class of σ and the order of the centralizer of σ in S_6.
   (b) Determine if $C_{S_6}(\sigma)$ is abelian or non-abelian. Prove your answer.

4. Let G be a finite group. Show that if G has a normal subgroup N of order 3 that is not contained in the center of G, then G has a subgroup of index 2. [Hint: The group G acts on N by conjugation.]

5. Show that a group of order 96 must have a normal subgroup of order 16 or 32.

III. Ring Theory

1. Let R be the ring of all 2×2 matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ where a and b are real numbers. Prove that R is isomorphic to $\mathbb{C}$, the field of complex numbers.

2. Let R be a commutative ring with identity. Show that if x and y are nilpotent elements of R then $x + y$ is nilpotent and the set of all nilpotent elements is an ideal in R.

3. Show that the ring $\mathbb{Z}[i]$ of Gaussian integers is a Euclidean ring and compute the greatest common divisor of $5 + i$ and 13 using the Euclidean algorithm.

4. Show that if R is a commutative ring with 1 that is not a field, then $R[x]$ is not a principal ideal ring.

5. Let R be an integral domain, S a multiplicative set, and let $S^{-1}R = \{\frac{r}{s} \mid r \in R, s \in S\}$ (contained in the field of fractions of R). Show that if P is a prime ideal of R, then $S^{-1}P$ is either a prime ideal of $S^{-1}R$ or else equals $S^{-1}R$.

IV. Field Theory

1. Find the minimal polynomial of $\alpha = \sqrt{3 + \sqrt{7}}$ over the field $\mathbb{Q}$ of rational numbers, and *prove* it is the minimal polynomial.

2. Show that if K is algebraic over F and $\sigma : K \to K$ is an F-monomorphism, then σ is onto.

3. Prove that the multiplicative group of a finite field must be cyclic.

4. Let F be any field and let $f(x) = x^n - 1 \in F[x]$. Show that if K is the splitting field of $f(x)$ over F, then K is separable over F (hence Galois) and that $\text{Gal}(K/F)$ is abelian.

5. (a) Find the Galois group of $x^3 - 2$ over $\mathbb{Q}$ and demonstrate the Galois correspondence between the subgroups of the Galois group and the subfields of the splitting field.

   (b) Find all automorphisms of $\mathbb{Q}(\sqrt[3]{2})$. Is there an $f(x) \in \mathbb{Q}[x]$ with splitting field $\mathbb{Q}(\sqrt[3]{2})$? Explain.