QUALIFYING EXAM IN ALGEBRA

August 2008

1. There are 18 problems on the exam. Work and turn in 10 problems, in the following categories.

I.	Linear Algebra	—	1 problem
II.	Group Theory	—	3 problems
III.	Ring Theory	—	2 problems
IV.	Field Theory	—	3 problems
Any of the four areas		—	1 problem

2. Turn in only 10 problems. No credit will be given for extra problems. All problems are weighted equally.

3. Put each problem on a separate sheet of paper, and write only on one side. Put your name on each page.

4. If you feel there is a misprint or error in the statement of a problem, then interpret it in such a way that the problem is not trivial.

I. Linear Algebra

1. Let A and B be nonsingular $n \times n$ matrices over $\mathbb{C}$.
   (a) Show that if $A^{-1}B^{-1}AB = cI$, $c \in \mathbb{C}$, then $c^n = 1$.
   (b) Show that if $AB - BA = cI$, $c \in \mathbb{C}$, then $c = 0$.

2. Let $B = \begin{bmatrix} 2 & -1 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.
   (a) Find the characteristic polynomial of B.
   (b) Find the minimal polynomial of B.
   (c) Find the eigenvalues of B.
   (d) Find the dimensions of all eigenspaces of B.
   (e) Find the Jordan canonical form of B.

3. Let V be a vector space over a field F. A linear transformation $T : V \to V$ is said to be *idempotent* if $T^2 = T$. Prove that if T is idempotent then $V = V_0 \oplus V_1$, where $T(v_0) = 0$ for all $v_0 \in V_0$ and $T(v_1) = v_1$ for all $v_1 \in V_1$.

II. Group Theory

1. Show that if G is a nonabelian finite group, then $|Z(G)| \leqslant \frac{1}{4}|G|$.

2. Let $G = A \times B$ be a direct product of the subgroups A and B. Suppose H is a subgroup of G that satisfies $HA = G = HB$ and $H \cap A = \langle 1 \rangle = H \cap B$. Prove that A is isomorphic to B.

3. (a) Prove that the additive group of the rational numbers is not cyclic.

   (b) Prove that a finitely generated subgroup of the additive group of the rational numbers must be cyclic.

4. Let G be a finite group and let P be a Sylow p-subgroup of G. Prove the following.

   (a) If M is any normal p-subgroup of G then M is a subgroup of P.

   (b) There is a normal p-subgroup N of G that contains all normal p-subgroups of G.

5. Let G be a group of order 168 and let P be a Sylow 7-subgroup of G. Show that either P is a normal subgroup of G or else the normalizer of P is a maximal subgroup of G.

III. Ring Theory

1. A ring R is called simple if $R^2 \neq 0$ and 0 and R are its only ideals. Show that the center of a simple ring is 0 or a field.

2. Let R be a commutative ring with identity. Suppose that for every $a \in R$, either a or $1 - a$ is invertible. Prove that $N = \{a \in R \mid a \text{ is not invertible}\}$ is an ideal of R.

3. Let $D = \mathbb{Z}(\sqrt{13}) = \{\, m + n\sqrt{13} \mid m, n \in \mathbb{Z} \,\}$ and $F = \mathbb{Q}(\sqrt{13})$, the field of fractions of D. Show the following:
   (a) $x^2 + 3x - 1$ is irreducible in $D[x]$ but not in $F[x]$.

   (b) D is not a unique factorization domain.

4. Let $F[x, y]$ be the polynomial ring over a field F in two indeterminates x, y. Show that the ideal generated by $\{x, y\}$ is not a principal ideal.

5. Let R be an integral domain, S a nonempty subset of R closed under multiplication, and let $S^{-1}R = \{\frac{r}{s} \,|\, r \in R,\, s \in S\}$ (contained in the field of fractions of R). Show that if P is a prime ideal of R then, $S^{-1}P$ is either a prime ideal of $S^{-1}R$ or else is equal to $S^{-1}R$.

IV. Field Theory

1. Let $f(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{Q}[x]$ be an irreducible polynomial of degree greater than 1 in which all roots lie on the unit circle of $\mathbb{C}$. Prove that $a_i = a_{n-i}$ for all i.

2. Find the minimal polynomial of $\alpha = \sqrt[3]{2+\sqrt{2}}$ over the field $\mathbb{Q}$ of rational numbers, and *prove* it is the minimal polynomial.

3. Let $f(x) \in F[x]$ be a polynomial, and let $f'(x)$ denote its formal derivative in $F[x]$. Prove that $f(x)$ has distinct roots in any extension field of F if and only if $f(x)$ and $f'(x)$ are relatively prime.

4. Let K be a splitting field for $x^5 - 2$ over $\mathbb{Q}$.
   (a) Determine $[K : \mathbb{Q}]$.
   (b) Show that $\mathrm{Gal}(K/\mathbb{Q})$ is non-abelian.
   (c) Find all normal intermediate extensions F and express as $F = \mathbb{Q}(\alpha)$ for appropriate α.

5. Let p be a prime. Show that the field of p^a elements is a subfield of the field of p^b elements if and only if $a|b$.