QUALIFYING EXAM IN ALGEBRA
August 2008

1. There are 18 problems on the exam. Work and turn in 10 problems, in the following categories.

   I. Linear Algebra — 1 problem
   II. Group Theory — 3 problems
   III. Ring Theory — 2 problems
   IV. Field Theory — 3 problems
   Any of the four areas — 1 problem

2. Turn in only 10 problems. No credit will be given for extra problems. All problems are weighted equally.

3. Put each problem on a separate sheet of paper, and write only on one side. Put your name on each page.

4. If you feel there is a misprint or error in the statement of a problem, then interpret it in such a way that the problem is not trivial.
I. Linear Algebra

1. Let $A$ and $B$ be nonsingular $n \times n$ matrices over $\mathbb{C}$.
   (a) Show that if $A^{-1}B^{-1}AB = cI$, $c \in \mathbb{C}$, then $c^n = 1$.
   (b) Show that if $AB - BA = cI$, $c \in \mathbb{C}$, then $c = 0$.

2. Let $B = \begin{bmatrix} 2 & -1 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.
   (a) Find the characteristic polynomial of $B$.
   (b) Find the minimal polynomial of $B$.
   (c) Find the eigenvalues of $B$.
   (d) Find the dimensions of all eigenspaces of $B$.
   (e) Find the Jordan canonical form of $B$.

3. Let $V$ be a vector space over a field $F$. A linear transformation $T : V \to V$ is said to be idempotent if $T^2 = T$. Prove that if $T$ is idempotent then $V = V_0 \oplus V_1$, where $T(v_0) = 0$ for all $v_0 \in V_0$ and $T(v_1) = v_1$ for all $v_1 \in V_1$.
II. Group Theory

1. Show that if $G$ is a nonabelian finite group, then $|Z(G)| \leq \frac{1}{4}|G|$.

2. Let $G = A \times B$ be a direct product of the subgroups $A$ and $B$. Suppose $H$ is a subgroup of $G$ that satisfies $HA = G = HB$ and $H \cap A = \langle 1 \rangle = H \cap B$. Prove that $A$ is isomorphic to $B$.

3. (a) Prove that the additive group of the rational numbers is not cyclic.
   (b) Prove that a finitely generated subgroup of the additive group of the rational numbers must be cyclic.

4. Let $G$ be a finite group and let $P$ be a Sylow $p$-subgroup of $G$. Prove the following.
   (a) If $M$ is any normal $p$-subgroup of $G$ then $M$ is a subgroup of $P$.
   (b) There is a normal $p$-subgroup $N$ of $G$ that contains all normal $p$-subgroups of $G$.

5. Let $G$ be a group of order 168 and let $P$ be a Sylow 7-subgroup of $G$. Show that either $P$ is a normal subgroup of $G$ or else the normalizer of $P$ is a maximal subgroup of $G$. 
III. Ring Theory

1. A ring $R$ is called simple if $R^2 \neq 0$ and 0 and $R$ are its only ideals. Show that the center of a simple ring is 0 or a field.

2. Let $R$ be a commutative ring with identity. Suppose that for every $a \in R$, either $a$ or $1 - a$ is invertible. Prove that $N = \{a \in R \mid a \text{ is not invertible}\}$ is an ideal of $R$.

3. Let $D = \mathbb{Z}(\sqrt{13}) = \{m + n\sqrt{13} \mid m, n \in \mathbb{Z}\}$ and $F = \mathbb{Q}(\sqrt{13})$, the field of fractions of $D$. Show the following:
   (a) $x^2 + 3x - 1$ is irreducible in $D[x]$ but not in $F[x]$.
   (b) $D$ is not a unique factorization domain.

4. Let $F[x, y]$ be the polynomial ring over a field $F$ in two indeterminates $x, y$. Show that the ideal generated by $\{x, y\}$ is not a principal ideal.

5. Let $R$ be an integral domain, $S$ a nonempty subset of $R$ closed under multiplication, and let $S^{-1}R = \{\frac{r}{s} \mid r \in R, s \in S\}$ (contained in the field of fractions of $R$). Show that if $P$ is a prime ideal of $R$ then, $S^{-1}P$ is either a prime ideal of $S^{-1}R$ or else is equal to $S^{-1}R$. 

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IV. Field Theory

1. Let \( f(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{Q}[x] \) be an irreducible polynomial of degree greater than 1 in which all roots lie on the unit circle of \( \mathbb{C} \). Prove that \( a_i = a_{n-i} \) for all \( i \).

2. Find the minimal polynomial of \( \alpha = \sqrt[3]{2} + \sqrt{2} \) over the field \( \mathbb{Q} \) of rational numbers, and prove it is the minimal polynomial.

3. Let \( f(x) \in F[x] \) be a polynomial, and let \( f'(x) \) denote its formal derivative in \( F[x] \). Prove that \( f(x) \) has distinct roots in any extension field of \( F \) if and only if \( f(x) \) and \( f'(x) \) are relatively prime.

4. Let \( K \) be a splitting field for \( x^5 - 2 \) over \( \mathbb{Q} \).
   (a) Determine \( [K : \mathbb{Q}] \).
   (b) Show that \( \text{Gal}(K/\mathbb{Q}) \) is non-abelian.
   (c) Find all normal intermediate extensions \( F \) and express as \( F = \mathbb{Q}(\alpha) \) for appropriate \( \alpha \).

5. Let \( p \) be a prime. Show that the field of \( p^a \) elements is a subfield of the field of \( p^b \) elements if and only if \( a | b \).