QUALIFYING EXAM IN ALGEBRA
August 2010

1. There are 18 problems on the exam. Work and turn in 10 problems, in the following categories.

   I. Linear Algebra — 1 problem
   II. Group Theory — 3 problems
   III. Ring Theory — 2 problems
   IV. Field Theory — 3 problems
   Any of the four areas — 1 problem

2. Turn in only 10 problems. No credit will be given for extra problems. All problems are weighted equally.

3. Put each problem on a separate sheet of paper, and write only on one side. Put your name on each page.

4. If you feel there is a misprint or error in the statement of a problem, then interpret it in such a way that the problem is not trivial.
I. Linear Algebra

1. Let \( A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ 0 & 0 & 3 \end{bmatrix} \) be a matrix over the field \( F \), where \( F \) is either the field of rational numbers or the field of \( p \) elements for some prime \( p \).

   (a) Find a basis of eigenvectors for \( A \) over those fields for which such a basis exists.

   (b) What is the Jordan canonical form of \( A \) over the fields of prime order not included in part (a)?

2. Let \( U, V \) and \( W \) be finite dimensional vector spaces with \( U \) a subspace of \( V \). Show that if \( T : V \rightarrow W \) is a linear transformation having the same rank as \( T|_U : U \rightarrow W \), then \( U \) is complemented in \( V \) by a subspace \( K \) satisfying \( T(x) = 0 \) for all \( x \in K \).

3. Let \( \{v_1, v_2, \ldots, v_n\} \) be a basis for a vector space \( V \) over \( \mathbb{R} \). Show that if \( w \) is any vector in \( V \), then for some choice of sign \( \pm \), \( \{v_1 \pm w, v_2, \ldots, v_n\} \) is a basis for \( V \).

II. Group Theory

1. Let \( H \) be a subgroup of the group \( G \). Show that the following are equivalent:

   (i) \( x^{-1}y^{-1}xy \in H \) for all \( x, y \in G \)

   (ii) \( H \trianglelefteq G \) and \( G/H \) is abelian.

2. Let \( G \) be a finite group and \( p \) a prime. Show that the intersection of all Sylow \( p \)-subgroups of \( G \) is a normal subgroup of \( G \).

3. Let \( G \) be a subgroup of the symmetric group \( S_n \). Show that if \( G \) contains an odd permutation then \( G \cap A_n \) is of index 2 in \( G \).

4. Let \( p \) be a prime and let \( G \) be a group of order \( p^n \) satisfying the following property:

   (*) If \( A \) and \( B \) are subgroups of \( G \) then \( A \trianglelefteq B \) or \( B \trianglelefteq A \).

   Prove that \( G \) is a cyclic group.

   [Note: This statement is also true without the assumption that \( G \) is a \( p \)-group.]

5. Let \( G \) be a finite simple group containing an element of order 9. Show that every proper subgroup of \( G \) has index at least 9.

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III. Ring Theory

1. Let $R$ be a commutative ring with 1. Show that an ideal $P$ of $R$ is prime if and only if $R/P$ is an integral domain.

2. The Jacobson Radical $J(R)$ of a ring $R$ is defined to be the intersection of all maximal ideals of $R$. Let $R$ be a commutative ring with 1 and let $x \in R$. Show that $x \in J(R)$ if and only if $1 - xy$ is a unit for all $y$ in $R$.

3. Let $R$ be a commutative ring with 1 in which every ideal is a prime ideal. Prove that $R$ is a field. [Hint: For $a \neq 0$ consider the ideals $(a)$ and $(a^2)$].

4. Let $D = \mathbb{Z}(\sqrt{5}) = \{m + n\sqrt{5} \mid m, n \in \mathbb{Z}\}$ — a subring of the field of real numbers and necessarily an integral domain (you need not show this) — and $F = \mathbb{Q}(\sqrt{5})$ its field of fractions. Show the following:
   (a) $x^2 + x - 1$ is irreducible in $D[x]$ but not in $F[x]$.
   (b) $D$ is not a unique factorization domain.

5. Let $D$ be an integral domain and let $c$ be an irreducible element in $D$. Show that the ideal $(x, c)$ generated by $x$ and $c$ in the polynomial ring $D[x]$ is not a principal ideal.
IV. Field Theory

1. Let \( K \) be a field extension of \( F \) of degree \( n \) and let \( f(x) \in F[x] \) be an irreducible polynomial of degree \( m > 1 \). Show that if \( m \) is relatively prime to \( n \), then \( f \) has no root in \( K \).

2. Let \( K \) be an extension field of \( F \) and let \( \alpha \) be an element of \( K \). Show that the following are equivalent:
   (i) \( \alpha \) is algebraic over \( F \),
   (ii) \( F(\alpha) \) is a finite dimensional extension of \( F \),
   (iii) \( \alpha \) is contained in a finite dimensional extension of \( F \).

3. Show that if \( F \) is a field of characteristic 0 then every algebraic extension of \( F \) is separable.

4. Let \( K \) be a finite Galois extension of \( F \) of characteristic 0. Show that if \( \text{Gal}(K/F) \) is a non-trivial 2-group, then there is a quadratic extension of \( F \) contained in \( K \).

5. Let \( u = \sqrt{2} + \sqrt{2} \), \( v = \sqrt{2} - \sqrt{2} \), and \( E = \mathbb{Q}(u) \), where \( \mathbb{Q} \) is the field of rational numbers.
   (a) Find the minimal polynomial \( f(x) \) of \( u \) over \( \mathbb{Q} \).
   (b) Show \( v \in E \). Hence conclude that \( E \) is a splitting field of \( f(x) \) over \( \mathbb{Q} \).
   (c) Determine the Galois group of \( E \) over \( \mathbb{Q} \).