QUALIFYING EXAM IN ALGEBRA
August 2013

1. There are 18 problems on the exam. Work and turn in 10 problems, in the following categories.

   I. Linear Algebra    —  1 problem
   II. Group Theory     —  3 problems
   III. Ring Theory     —  2 problems
   IV. Field Theory     —  3 problems
   Any of the four areas —  1 problem

2. Turn in only 10 problems. No credit will be given for extra problems. All problems are weighted equally.

3. Put each problem on a separate sheet of paper, and write only on one side. Put your name on each page.

4. If you feel there is a misprint or error in the statement of a problem, then interpret it in such a way that the problem is not trivial.
I. Linear Algebra

1. Let $V$ be a vector space and let $T : V \to V$ be a linear transformation.

(a) Show that $T$ is invertible if and only if the minimal polynomial of $T$ has non-zero constant term.

(b) Show that if $T$ is invertible then $T^{-1}$ can be expressed as a polynomial in $T$.

2. Let $V$ and $W$ be finite dimensional vector spaces and let $T : V \to W$ be a linear transformation. Show that $\dim(\ker T) + \dim(\text{Im } T) = \dim(V)$.

3. Let $V$ be the vector space of $n \times n$ matrices over the field $\mathbb{R}$ of real numbers. Let $U$ be the subspace of $V$ consisting of symmetric matrices and $W$ the subspace of $V$ consisting of skew-symmetric matrices. Show that $V = U \oplus W$.

II. Group Theory

1. Let $G$ be a group acting transitively on a set $\Omega$. Show that the following are equivalent.

(i) The action is doubly transitive (i.e., for any two ordered pairs $(\alpha_1, \beta_1), (\alpha_2, \beta_2)$ of elements of $\Omega$ with $\alpha_1 \neq \beta_1$ and $\alpha_2 \neq \beta_2$, there is an element $g$ in $G$ such that $g \cdot \alpha_1 = \alpha_2$ and $g \cdot \beta_1 = \beta_2$).

(ii) For all $\alpha \in \Omega$, the stabilizer $G_\alpha$ acts transitively on $\Omega - \{\alpha\}$.

2. Show that if $G$ is a subgroup of $S_n$ of index 2, then $G = A_n$.

3. Let $G$ be a finite group and let $P$ be a Sylow $p$-subgroup of $G$. Prove the following.

(a) If $M$ is any normal $p$-subgroup of $G$ then $M \leq P$.

(b) There is a normal $p$-subgroup $N$ of $G$ that contains all normal $p$-subgroups of $G$.

4. Let $G$ be a group and $G'$ its commutator subgroup. Show that if $g \in G$, then the conjugacy class of $g$ is contained in $gG'$.

5. Show that every group of order $2013 = 3 \cdot 11 \cdot 61$ has a cyclic normal subgroup of index 3.
III. Ring Theory

1. Let $R$ be a commutative ring with 1 and let $I$ and $J$ be ideals of $R$ such that $I + J = R$. Show that $R/(I \cap J) \cong R/I \oplus R/J$.

2. Let $R$ be a non-zero commutative ring with 1. Show that an ideal $M$ of $R$ is maximal if and only if $R/M$ is a field.

3. Let $R$ be a commutative ring with 1. An ideal $I$ of $R$ is called a primary ideal if $I \neq R$ and for all $x, y \in R$ with $xy \in I$, either $x \in I$ or $y^n \in I$ for some integer $n \geq 1$.
   
   (a) Show that an ideal $I$ of $R$ is primary if and only if $R/I \neq 0$ and every zero-divisor in $R/I$ is nilpotent.
   
   (b) Show that if $I$ is a primary ideal of $R$ then the radical $\text{Rad}(I)$ of $I$ is a prime ideal. (Recall that $\text{Rad}(I) = \{x \in R \mid x^n \in I \text{ for some } n\}$.)

4. Let $D$ be an integral domain and $F$ a subring of $D$ that is a field. Show that if each element of $D$ is algebraic over $F$, then $D$ is a field.

5. Let $R$ be a commutative ring with $1 \neq 0$ in which the set of nonunits is closed under addition. Prove that $R$ is local, i.e., has a unique maximal ideal.
IV. Field Theory

1. Let $K$ be an extension field of the field $F$ such that $[K : F]$ is odd. Show that if $u \in K$ then $F(u) = F(u^2)$.

2. Let $F$ be a field and let $f(x) \in F[x]$ have splitting field $K$. Show that if the degree of $f$ is a prime $p$ and $[K : F] = tp$ for some integer $t$, then
   
   (a) $f(x)$ is irreducible over $F$ and
   
   (b) if $t > 1$ then $K$ is a separable extension of $F$.


4. Let $\alpha = \sqrt{5} + 2\sqrt{5}$. Show that $\mathbb{Q}(\alpha)$ is a cyclic Galois extension of $\mathbb{Q}$ of degree 4. Find all fields $F$ with $\mathbb{Q} \subseteq F \subseteq \mathbb{Q}(\alpha)$.
   
   [Hint: Show that $f(x) = x^4 - 10x^2 + 5$ is the minimal polynomial of $\alpha$ over $\mathbb{Q}$ and that the roots of $f$ are $\pm \alpha, \pm \frac{\sqrt{5}}{\alpha}$]

5. Let $E$ and $F$ be subfields of a finite field $K$. Show that if $E$ is isomorphic to $F$ then $E = F$. 