QUALIFYING EXAM IN ALGEBRA

August 2013

1. There are 18 problems on the exam. Work and turn in 10 problems, in the following categories.

I.	Linear Algebra	—	1 problem
II.	Group Theory	—	3 problems
III.	Ring Theory	—	2 problems
IV.	Field Theory	—	3 problems
Any of the four areas		—	1 problem

2. Turn in only 10 problems. No credit will be given for extra problems. All problems are weighted equally.

3. Put each problem on a separate sheet of paper, and write only on one side. Put your name on each page.

4. If you feel there is a misprint or error in the statement of a problem, then interpret it in such a way that the problem is not trivial.

I. Linear Algebra

1. Let V be a vector space and let $T : V \to V$ be a linear transformation.

   (a) Show that T is invertible if and only if the minimal polynomial of T has non-zero constant term.

   (b) Show that if T is invertible then T^{-1} can be expressed as a polynomial in T.

2. Let V and W be finite dimensional vector spaces and let $T : V \to W$ be a linear transformation. Show that $\dim(\ker T) + \dim(\mathrm{Im}\ T) = \dim(V)$.

3. Let V be the vector space of $n \times n$ matrices over the field $\mathbb{R}$ of real numbers. Let U be the subspace of V consisting of symmetric matrices and W the subspace of V consisting of skew-symmetric matrices. Show that $V = U \oplus W$.

II. Group Theory

1. Let G be a group acting *transitively* on a set Ω. Show that the following are equivalent.

   (i) The action is doubly transitive (i.e., for any two ordered pairs (α_1, β_1), (α_2, β_2) of elements of Ω with $\alpha_1 \neq \beta_1$ and $\alpha_2 \neq \beta_2$, there is an element g in G such that $g \cdot \alpha_1 = \alpha_2$ and $g \cdot \beta_1 = \beta_2$).

   (ii) For all $\alpha \in \Omega$, the stabilizer G_α acts transitively on $\Omega - \{\alpha\}$.

2. Show that if G is a subgroup of S_n of index 2, then $G = A_n$.

3. Let G be a finite group and let P be a Sylow p-subgroup of G. Prove the following.

   (a) If M is any normal p-subgroup of G then $M \leqslant P$.

   (b) There is a normal p-subgroup N of G that contains all normal p-subgroups of G.

4. Let G be a group and G' its commutator subgroup. Show that if $g \in G$, then the conjugacy class of g is contained in gG'.

5. Show that every group of order $2013 = 3 \cdot 11 \cdot 61$ has a cyclic normal subgroup of index 3.

III. Ring Theory

1. Let R be a commutative ring with 1 and let I and J be ideals of R such that $I+J=R$. Show that $R/(I \cap J) \cong R/I \oplus R/J$.

2. Let R be a non-zero commutative ring with 1. Show that an ideal M of R is maximal if and only if R/M is a field.

3. Let R be a commutative ring with 1. An ideal I of R is called a *primary* ideal if $I \neq R$ and for all $x, y \in R$ with $xy \in I$, either $x \in I$ or $y^n \in I$ for some integer $n \geqslant 1$.

   (a) Show that an ideal I of R is primary if and only if $R/I \neq 0$ and every zero-divisor in R/I is nilpotent.

   (b) Show that if I is a primary ideal of R then the radical $\mathrm{Rad}(I)$ of I is a prime ideal. (Recall that $\mathrm{Rad}(I) = \{x \in R \mid x^n \in I \text{ for some } n\}$.)

4. Let D be an integral domain and F a subring of D that is a field. Show that if each element of D is algebraic over F, then D is a field.

5. Let R be a commutative ring with $1 \neq 0$ in which the set of nonunits is closed under addition. Prove that R is local, i.e., has a unique maximal ideal.

IV. Field Theory

1. Let K be an extension field of the field F such that $[K : F]$ is odd. Show that if $u \in K$ then $F(u) = F(u^2)$.

2. Let F be a field and let $f(x) \in F[x]$ have splitting field K. Show that if the degree of f is a prime p and $[K : F] = tp$ for some integer t, then

   (a) $f(x)$ is irreducible over F and

   (b) if $t > 1$ then K is a separable extension of F.

3. Let K be a Galois extension of F with $|\mathrm{Gal}(K/F)| = 12$. Prove that there exists a subfield E of K containing F with $[E : F] = 3$. Does a subextension L necessarily exist satisfying $[L : F] = 2$? Explain.

4. Let $\alpha = \sqrt{5 + 2\sqrt{5}}$. Show that $\mathbb{Q}(\alpha)$ is a cyclic Galois extension of $\mathbb{Q}$ of degree 4. Find all fields F with $\mathbb{Q} \subseteq F \subseteq \mathbb{Q}(\alpha)$.
   [Hint: Show that $f(x) = x^4 - 10x^2 + 5$ is the minimal polynomial of α over $\mathbb{Q}$ and that the roots of f are $\pm\alpha$, $\pm\frac{\sqrt{5}}{\alpha}$.]

5. Let E and F be subfields of a finite field K. Show that if E is isomorphic to F then $E = F$.