QUALIFYING EXAM IN ALGEBRA

August 2014

1. There are 18 problems on the exam. Work and turn in 10 problems, in the following categories.

I.	Linear Algebra	—	1 problem
II.	Group Theory	—	3 problems
III.	Ring Theory	—	2 problems
IV.	Field Theory	—	3 problems
Any of the four areas		—	1 problem

2. Turn in only 10 problems. No credit will be given for extra problems. All problems are weighted equally.

3. Put each problem on a separate sheet of paper, and write only on one side. Put your name on each page.

4. If you feel there is a misprint or error in the statement of a problem, then interpret it in such a way that the problem is not trivial.

I. Linear Algebra

1. A matrix A has characteristic polynomial $\Delta(x) = (x-3)^5$ and minimal polynomial $m(x) = (x-3)^3$.

   (a) List all possible Jordan canonical forms for A.

   (b) Determine the Jordan canonical form of the matrix

$$A = \begin{bmatrix} 3 & -1 & 2 & 0 & 0 \\ 2 & 3 & 0 & -2 & 0 \\ 1 & 0 & 3 & -1 & 0 \\ 0 & -1 & 2 & 3 & 0 \\ 0 & 2 & -3 & 0 & 3 \end{bmatrix}$$

   which has the given characteristic and minimal polynomials.

2. A linear transformation $T : V \to W$ is said to be independence preserving if $T(I) \subseteq W$ is linearly independent whenever $I \subseteq V$ is a linearly independent set. Show that T is independence preserving if and only if T is one-to-one.

3. Let V be the vector space over the field $\mathbb{R}$ of real numbers consisting of all functions from $\mathbb{R}$ into $\mathbb{R}$. Let U be the subspace of even functions and W the subspace of odd functions. Show that $V = U \oplus W$.

II. Group Theory

1. Let G be a group acting on the set S and let H be a subgroup of G acting transitively on S. Show that if $t \in S$ then $G = G_t H$, where G_t is the stabilizer of t in G.

2. Let N be a normal subgroup of G. Show that if $N \cap G' = \langle 1 \rangle$, then N is contained in the center of G.

3. Let n be an integer and p a prime dividing n. Assume that there exists exactly one divisor d of n satisfying both $d > 1$ and $d \equiv 1 (\text{mod } p)$. Prove that if G is any finite group of order n and P is a Sylow p-subgroup of G, then either $P \trianglelefteq G$ or else $N_G(P)$ is a maximal subgroup of G.

4. Let G be a finite group and let M be a maximal subgroup of G.
   Let $Z(G)$ denote the center of G, G' the commutator subgroup of G, and $\Phi(G)$ the Frattini subgroup of G, i.e., the intersection of all maximal subgroups of G.

   (a) Show that if $Z(G)$ is not contained in M, then $M \trianglelefteq G$.

   (b) Show that either $Z(G) \leqslant M$ or $G' \leqslant M$.

   (c) Show that $Z(G) \cap G' \leqslant \Phi(G)$.

5. Show that a group of order $380 = 2^2 \cdot 5 \cdot 19$ must be solvable.

III. Ring Theory

1. Show that if R is a finite commutative ring with identity then every prime ideal of R is a maximal ideal.

2. Let R be a non-zero commutative ring with 1. Show that if I is an ideal of R such that $1+a$ is a unit in R for all $a \in I$, then I is contained in every maximal ideal of R.

3. Find all values of a in $\mathbb{Z}_5$ such that the quotient ring

$$\mathbb{Z}_5[x]/(x^3 + 2x^2 + ax + 3)$$

is a field. Justify your answer.

4. Let $D = \mathbb{Z}(\sqrt{-11}) = \{m+n\sqrt{-11} \mid m, n \in \mathbb{Z}\}$ and $F = \mathbb{Q}(\sqrt{-11})$ its field of fractions. Show the following:

   (a) $x^2 - x + 3$ is irreducible in $D[x]$ but not in $F[x]$.

   (b) D is not a unique factorization domain.

5. Let R be a commutative ring with 1 and D a multiplicative subset of R containing 1. Let J be an ideal in the ring of fractions $D^{-1}R$ and let

$$I = \left\{ a \in R \;\middle|\; \frac{a}{d} \in J \text{ for some } d \in D \right\}.$$

Show that I is an ideal of R.

IV. Field Theory

1. Let K be an extension field of F and let α be an element of K. Show that if $F(\alpha) = F(\alpha^2)$, then α is algebraic over F.

2. Let $K = F(u)$ be a separable extension of F with $u^m \in F$ for some positive integer m. Show that if the characteristic of F is p and $m = p^t r$, then $u^r \in F$.

3. Suppose $K = F(\alpha)$ is a proper Galois extension of F and assume there exists an element σ of $\mathrm{Gal}(K/F)$ satisfying $\sigma(\alpha) = \alpha^{-1}$. Show that $[K : F]$ is even and that $[F(\alpha + \alpha^{-1}) : F] = \frac{1}{2}[K : F]$.

4. Let $f(x) = x^4 + ax^3 + bx^2 + ax + 1 \in \mathbb{Q}[x]$ and let F be a splitting field over $\mathbb{Q}$. Show that if α is a root of f then $1/\alpha$ is also a root, and $|\mathrm{Gal}(F/\mathbb{Q})| \leqslant 8$.

5. Prove that any finite extension of a finite field must be a simple extension.