QUALIFYING EXAM IN ALGEBRA
August 2014

1. There are 18 problems on the exam. Work and turn in 10 problems, in the following categories.

   I. Linear Algebra    — 1 problem
   II. Group Theory    — 3 problems
   III. Ring Theory    — 2 problems
   IV. Field Theory    — 3 problems
   Any of the four areas — 1 problem

2. Turn in only 10 problems. No credit will be given for extra problems. All problems are weighted equally.

3. Put each problem on a separate sheet of paper, and write only on one side. Put your name on each page.

4. If you feel there is a misprint or error in the statement of a problem, then interpret it in such a way that the problem is not trivial.
I. Linear Algebra

1. A matrix $A$ has characteristic polynomial $\Delta(x) = (x - 3)^5$ and minimal polynomial $m(x) = (x - 3)^3$.

   (a) List all possible Jordan canonical forms for $A$.

   (b) Determine the Jordan canonical form of the matrix

   $$A = \begin{bmatrix}
   3 & -1 & 2 & 0 & 0 \\
   2 & 3 & 0 & -2 & 0 \\
   1 & 0 & 3 & -1 & 0 \\
   0 & -1 & 2 & 3 & 0 \\
   0 & 2 & -3 & 0 & 3 \\
   \end{bmatrix}$$

   which has the given characteristic and minimal polynomials.

2. A linear transformation $T : V \rightarrow W$ is said to be independence preserving if $T(I) \subseteq W$ is linearly independent whenever $I \subseteq V$ is a linearly independent set. Show that $T$ is independence preserving if and only if $T$ is one-to-one.

3. Let $V$ be the vector space over the field $\mathbb{R}$ of real numbers consisting of all functions from $\mathbb{R}$ into $\mathbb{R}$. Let $U$ be the subspace of even functions and $W$ the subspace of odd functions. Show that $V = U \oplus W$. 
II. Group Theory

1. Let $G$ be a group acting on the set $S$ and let $H$ be a subgroup of $G$ acting transitively on $S$. Show that if $t \in S$ then $G = G_t H$, where $G_t$ is the stabilizer of $t$ in $G$.

2. Let $N$ be a normal subgroup of $G$. Show that if $N \cap G' = \langle 1 \rangle$, then $N$ is contained in the center of $G$.

3. Let $n$ be an integer and $p$ a prime dividing $n$. Assume that there exists exactly one divisor $d$ of $n$ satisfying both $d > 1$ and $d \equiv 1 \pmod{p}$. Prove that if $G$ is any finite group of order $n$ and $P$ is a Sylow $p$-subgroup of $G$, then either $P \leq G$ or else $N_G(P)$ is a maximal subgroup of $G$.

4. Let $G$ be a finite group and let $M$ be a maximal subgroup of $G$.
   Let $Z(G)$ denote the center of $G$, $G'$ the commutator subgroup of $G$, and $\Phi(G)$ the Frattini subgroup of $G$, i.e., the intersection of all maximal subgroups of $G$.
   
   (a) Show that if $Z(G)$ is not contained in $M$, then $M \trianglelefteq G$.
   
   (b) Show that either $Z(G) \leq M$ or $G' \leq M$.
   
   (c) Show that $Z(G) \cap G' \leq \Phi(G)$.

5. Show that a group of order $380 = 2^2 \cdot 5 \cdot 19$ must be solvable.
III. Ring Theory

1. Show that if $R$ is a finite commutative ring with identity then every prime ideal of $R$ is a maximal ideal.

2. Let $R$ be a non-zero commutative ring with 1. Show that if $I$ is an ideal of $R$ such that $1 + a$ is a unit in $R$ for all $a \in I$, then $I$ is contained in every maximal ideal of $R$.

3. Find all values of $a$ in $\mathbb{Z}_5$ such that the quotient ring

$$\mathbb{Z}_5[x]/(x^3 + 2x^2 + ax + 3)$$

is a field. Justify your answer.

4. Let $D = \mathbb{Z}(\sqrt{-11}) = \{m+n\sqrt{-11} \mid m, n \in \mathbb{Z}\}$ and $F = \mathbb{Q}(\sqrt{-11})$ its field of fractions. Show the following:

   (a) $x^2 - x + 3$ is irreducible in $D[x]$ but not in $F[x]$.

   (b) $D$ is not a unique factorization domain.

5. Let $R$ be a commutative ring with 1 and $D$ a multiplicative subset of $R$ containing 1. Let $J$ be an ideal in the ring of fractions $D^{-1}R$ and let

$$I = \left\{a \in R \left| \frac{a}{d} \in J \text{ for some } d \in D \right. \right\}.$$

Show that $I$ is an ideal of $R$. 

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IV. Field Theory

1. Let $K$ be an extension field of $F$ and let $\alpha$ be an element of $K$. Show that if $F(\alpha) = F(\alpha^2)$, then $\alpha$ is algebraic over $F$.

2. Let $K = F(u)$ be a separable extension of $F$ with $u^m \in F$ for some positive integer $m$. Show that if the characteristic of $F$ is $p$ and $m = p^r t$, then $u^r \in F$.

3. Suppose $K = F(\alpha)$ is a proper Galois extension of $F$ and assume there exists an element $\sigma$ of Gal$(K/F)$ satisfying $\sigma(\alpha) = \alpha^{-1}$. Show that $[K : F]$ is even and that $[F(\alpha + \alpha^{-1}) : F] = \frac{1}{2}[K : F]$.

4. Let $f(x) = x^4 + ax^3 + bx^2 + ax + 1 \in \mathbb{Q}[x]$ and let $F$ be a splitting field over $\mathbb{Q}$. Show that if $\alpha$ is a root of $f$ then $1/\alpha$ is also a root, and $|\text{Gal}(F/\mathbb{Q})| \leq 8$.

5. Prove that any finite extension of a finite field must be a simple extension.