# QUALIFYING EXAM IN ALGEBRA

August 2015

9:00 AM – 12:00 Noon

1. There are 18 problems on the exam. Work and turn in 10 problems, in the following categories.

| | | | |
|---|---|---|---|
| I. | Linear Algebra | – | 1 problem |
| II. | Group Theory | – | 3 problems |
| III. | Ring Theory | – | 3 problems |
| IV. | Field Theory | – | 3 problems |

2. Turn in only 10 problems. No credit will be given for extra problems. All problems are weighted equally.

3. Put each problem on a separate sheet of paper, and write only on one side. Put your name on each page.

4. If you feel there is a misprint or error in the statement of a problem, then interpret it in such a way that the problem is not trivial.

# I. Linear Algebra

1. Let $A$ and $B$ be $n \times n$ matrices over the complex numbers and assume $AB = BA$. Prove that $A$ and $B$ share a common eigenvector.

2. Let $T : V \to W$ be a surjective linear transformation. Assume that for all subsets $S \subseteq V$ that if $T(S)$ spans $W$ then $S$ spans $V$. Prove that $T$ is one-to-one.

3. The characteristic polynomial of a certain $4 \times 4$ matrix is known to have the two distinct roots 2 and 3, with the multiplicity of the root 3 no greater than the multiplicity of 2. List all the possible Jordan normal forms of this matrix. (A rearrangement of Jordan blocks of a Jordan normal form is not regarded as a new form.)

# II. Group Theory

1. Let $G$ be a group of order $pqr$ where $p < q < r$ are primes. Prove that some Sylow subgroup for one of these primes is normal.

2. Let Let $g \in S_n$ (the symmetric group on $n$ letters) be a product of two disjoint cycles, one a $k$-cycle and the other an $\ell$-cycle where $k < \ell$ and $k + \ell = n$. If $H = \mathbf{C}_{S_n}(g) = \{h \in S_n \mid hg = gh\}$ then prove that $H$ is not a transitive subgroup of $S_n$.

3. Let $G$ be a finite group and $P$ a Sylow $p$-subgroup of $G$. If $N \trianglelefteq G$ prove that $P \cap N$ is a Sylow $p$-subgroup of $N$.

4. Let $G$ be a finite group.
   (a) Show that every proper subgroup of $G$ is contained in a maximal subgroup.
   (b) Show that the intersection of all maximal subgroups of $G$ is a normal subgroup.

5. Let $p$ be a prime and let $G$ be a non-abelian group of order $p^3$.
   (a) Show that the center $Z(G)$ of $G$ and the commutator subgroup of $G$ are equal and of order $p$.
   (b) Show that $G/Z(G) \cong Z_p \times Z_p$.

# III. Ring Theory

1. Let $M_1 \neq M_2$ be two maximal ideals of the commutative ring $R$, and let $I = M_1 \cap M_2$. Prove that $R/I$ is isomorphic to the direct sum of two fields.

2. Let $R$ be a ring and $a, b \in R$ with $ab = 1$. Let $X = \{x \in R \mid ax = 1\}$.
   (a) If $x \in X$, prove $b + (1 - xa) \in X$.
   (b) If $\varphi : X \to X$ is the mapping given by $\varphi(x) = b + (1 - xa)$, then $\varphi$ is one-to-one.
   (c) If $X$ has more than one element, then $X$ is an infinite set.

3. Let $R$ be a commutative ring. Show that if $x$ and $y$ are nilpotent elements of $R$ then $x + y$ is nilpotent and the set of all nilpotent elements is an ideal in $R$.

4. Prove that a Euclidean domain is a principal ideal domain.

5. Let $R$ be a commutative Noetherian ring in which every 2-generated ideal is principal. Prove that $R$ is a principal ideal domain.

# IV. Field Theory

1. Let $f(x) = (x^2 - 2)(x^2 - 3)$. Find, with proof, all the elements of $\mathrm{Gal}(K/\mathbb{Q})$ where $K$ is the splitting field of $f(x)$ over $\mathbb{Q}$. Also, find all intermediate subfields.

2. Let $F$ be a field of characteristic 0. Suppose $a \in F$ and $f(x) = x^4 + ax^2 + 1$ is irreducible over $F$. Let $K$ be a splitting field of $f(x)$ over $F$. Prove that $\mathrm{Gal}(K/F)$ has order 4 and is not cyclic. (Hint: If $\alpha$ is a root then so are $-\alpha$ and $1/\alpha$.)

3. Let $\alpha$ belong to some field extension of the field $F$. Prove $F(\alpha) = F[\alpha]$ if and only if $\alpha$ is algebraic over $F$.

4. Show that $p(x) = x^3 + 2x + 1$ is irreducible in $\mathbb{Q}[x]$. Let $\theta$ be a root of $p(x)$ in an extension field. Find the inverse of $1 + \theta$ in $\mathbb{Q}[\theta]$.

5. Show that $p(x) = x^3 + x - 6$ is irreducible over $\mathbb{Q}(i)$.