QUALIFYING EXAM IN ALGEBRA

August 2017

1. There are 18 problems on the exam. Work and turn in 10 problems, in the following categories.

I.	Linear Algebra	—	1 problem
II.	Group Theory	—	3 problems
III.	Ring Theory	—	2 problems
IV.	Field Theory	—	3 problems
Any of the four areas		—	1 problem

2. Turn in only 10 problems. No credit will be given for extra problems. All problems are weighted equally.

3. Put each problem on a separate sheet of paper, and write only on one side. Put your name on each page.

4. If you feel there is a misprint or error in the statement of a problem, then interpret it in such a way that the problem is not trivial.

I. Linear Algebra

1. Let A be an $n \times n$ matrix, $\boldsymbol{v}$ a column vector, and suppose $\{\boldsymbol{v}, A\boldsymbol{v}, \ldots, A^{n-1}\boldsymbol{v}\}$ is linearly independent. Prove that if B is any matrix that commutes with A, then B is a polynomial in A.

2. (a) Show that two 3×3 complex matrices are similar if and only if they have the same characteristic and minimal polynomials.

   (b) Is the conclusion of part (a) true for larger matrices? Prove or give a counter-example.

3. Let U, V, and W be vector spaces over a field F and let $S : U \to V$ and $T : V \to W$ be linear transformations such that $T \circ S = \mathbf{0}$, the zero map. Show that

$$\dim(W/\mathrm{Im}\ T) - \dim(\ker T/\mathrm{Im}\ S) + \dim\ker S = \dim W - \dim V + \dim U.$$

II. Group Theory

1. Show that if H is a characteristic subgroup of N and N is a characteristic subgroup of G, then H is a characteristic subgroup of G.

2. Let H and K be subgroups of a group G, with $K \trianglelefteq G$ and $K \leqslant H$. Show that H/K is contained in the center of G/K if and only if $[H, G] \leqslant K$ (where $[H, G] = \langle h^{-1}g^{-1}hg \,|\, h \in H,\, g \in G\rangle$).

3. Let G be a simple group of order greater than 2 and let $\mathrm{Aut}(G)$ be its automorphism group. Show that the center of $\mathrm{Aut}(G)$ is trivial if and only if G is nonabelian.

4. Prove that a group G of order 36 must have a normal subgroup of order 3 or 9.

5. Show that if G is a nilpotent group and $\langle 1\rangle \neq N \trianglelefteq G$, then $N \cap Z(G) \neq \langle 1\rangle$.

III. Ring Theory

1. Denote the set of invertible elements of the ring $\mathbb{Z}_n$ by U_n.
   (a) List all the elements of U_{24}.

   (b) Is U_{24} a cyclic group under multiplication? Justify your answer.

2. Let m and n be relatively prime integers.
   (a) Show that if c and d are any integers, then there is an integer x such that $x \equiv c \pmod{m}$ and $x \equiv d \pmod{n}$.

   (b) Show that $\mathbb{Z}_{mn}$ and $\mathbb{Z}_m \times \mathbb{Z}_n$ are isomorphic as rings.

3. Let R be a commutative ring and P a prime ideal of R. Show that there is a prime ideal $P_0 \subseteq P$ that does not properly contain any prime ideal.

4. Show that if R is a commutative ring with 1 that is not a field, then $R[x]$ is not a principal ideal ring.

5. If R is any ring with identity, let $J(R)$ denote the Jacobson radical of R. Show that if e is any idempotent of R, then $J(eRe) = eJ(R)e$.

IV. Field Theory

1. Let F be a field and F^* its multiplicative group. Show that the abelian groups $(F, +)$ and $(F^*, \cdot)$ are not isomorphic.

2. Find the minimal polynomial of $\alpha = \sqrt{11 + \sqrt{3}}$ over the field $\mathbb{Q}$ of rational numbers, and *prove* it is the minimal polynomial.

3. Suppose E is an algebraic extension of F and $\bar{E}$ is an algebraic closure of E. Show that $\bar{E}$ is an algebraic closure of F.

4. Let K be the splitting field of $x^3 - 3$ over $\mathbb{Q}$. Use Galois Theory to identify $\mathrm{Gal}(K/\mathbb{Q})$ and find explicitly all of the intermediate subfields.

5. Show that any two finite fields of the same order are isomorphic.