QUALIFYING EXAM IN ALGEBRA

August 2020

1. There are 18 problems on the exam. Work and turn in 10 problems, in the following categories.

I.	Linear Algebra	—	1 problem
II.	Group Theory	—	3 problems
III.	Ring Theory	—	2 problems
IV.	Field Theory	—	3 problems
Any of the four areas		—	1 problem

2. Turn in only 10 problems. No credit will be given for extra problems. All problems are weighted equally.

3. Put each problem on a separate sheet of paper, and write only on one side. Put your name on each page.

4. If you feel there is a misprint or error in the statement of a problem, then interpret it in such a way that the problem is not trivial.

I. Linear Algebra

1. (a) Let F be an algebraically closed field and $n > 1$. If V is a subspace of $F^{n\times n}$ satisfying $\dim V \geq 2$, prove that V contains a nonzero singular matrix.

   (b) Show by example that (a) is false if F is not algebraically closed.

2. (a) Let $J = \begin{bmatrix} \lambda & 1 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \lambda & 1 \\ 0 & \dots & 0 & \lambda \end{bmatrix}$ be an $n \times n$ Jordan block. Show that J is similar to its transpose J^{T}.

   (b) If A is an $n \times n$ matrix with entries in $\mathbb{C}$, prove that A is similar to A^{T}.

3. Let $\{v_1, v_2, \dots, v_n\}$ be a basis for a vector space V over $\mathbb{R}$. Show that if w is any vector in V, then for some choice of sign $\pm$, $\{v_1 \pm w, v_2, \dots, v_n\}$ is a basis for V .

II. Group Theory

1. Let $n \geq 3$ be odd. Prove that the set of n-cycles in A_n is not a conjugacy class of A_n.

2. Let $K \leq H \lhd G$ and assume every automorphism of H is inner. Prove that $G = H\,N_G(K)$, where $N_G(K)$ is the normalizer of K in G.

3. Let P be a Sylow p-subgroup of the finite group G and let H be a subgroup containing the normalizer $N_G(P)$ of P. Prove that $N_G(H) = H$.

4. Show that if G is a group of order $392 = 2^3 \cdot 7^2$, then G has a normal subgroup of order 7 or a normal subgroup of order 49.

5. Let H be a subgroup of the group G with the property that whenever two elements of G are conjugate, then the conjugating element can be chosen within H. Prove that the commutator subgroup G' of G is contained in H.

III. Ring Theory

1. Let R be a commutative ring with unity and let I and J be ideals of R. Suppose that $I + J = R$. Prove $I \cdot J = I \cap J$.

2. Let R be a commutative ring with 1. Show that if I is an ideal of R such that $1 + a$ is a unit in R for all $a \in I$, then I is contained in every maximal ideal of R.

3. Prove that the set of nilpotent elements of a commutative ring R is contained in the intersection of all prime ideals of R.

4. (a) Show that $x^4 + x^3 + x^2 + x + 1$ is irreducible in $\mathbb{Z}_3[x]$.

   (b) Show that $x^4 + 1$ is not irreducible in $\mathbb{Z}_3[x]$.

5. Let R be an integral domain that satisfies the descending chain condition on ideals. That is, whenever $I_1 \supset I_2 \supset I_3 \supset \cdots$ is a descending chain of ideals of R, there exists $m \in \mathbb{N}$ such that $I_m = I_k$ for all $k \geq m$. Prove that R is a field.

IV. Field Theory

1. Let p and q be distinct primes. Prove $\sqrt{q}$ does not belong to $\mathbb{Q}(\sqrt{p})$.

2. Let F be a field, let $E = F(\alpha)$ be a simple extension field of F, and let $b \in E \setminus F$. Prove that α is algebraic over $F(\beta)$.

3. Show that if F is a field of characteristic 0 then every algebraic extension of F is separable. (Provide an argument, do not just say that every field of characteristic 0 is perfect and every algebraic extension of a perfect field is separable.)

4. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of degree 4 with exactly 2 real roots. Show that the Galois group of f over $\mathbb{Q}$ has order either 8 or 24.

5. Find the minimal polynomial of $\sqrt{3 + \sqrt{5}}$ over Q. Then find, with proof, the Galois group of the splitting field of this polynomial over $\mathbb{Q}$.