QUALIFYING EXAM IN ALGEBRA

August 2021

1. There are 18 problems on the exam. Work and turn in 10 problems, in the following categories.

I.	Linear Algebra	—	1 problem
II.	Group Theory	—	3 problems
III.	Ring Theory	—	2 problems
IV.	Field Theory	—	3 problems
Any of the four areas		—	1 problem

2. Turn in only 10 problems. No credit will be given for extra problems. All problems are weighted equally.

3. Put each problem on a separate sheet of paper, and write only on one side. Put your name on each page.

4. If you feel there is a misprint or error in the statement of a problem, then interpret it in such a way that the problem is not trivial.

I. Linear Algebra

1. (a) Prove that a 2×2 scalar matrix A over a field F has a square root (i.e., a matrix B satisfying $B^2 = A$).

   (b) Prove that a real symmetric matrix having the property that every negative eigenvalue occurs with even multiplicity has a square root. [Hint: Use (a).]

2. Let $T : V \to W$ be a surjective linear transformation of finite dimensional vector spaces over a field F (acting on the left). Show that there is a linear transformation $S : W \to V$ such that $T \circ S$ is the identity map on W.

3. Let V be a vector space over the field F. Let V^* be the dual space of V and let V^{**} be the dual space of V^*. Show that there is an injective linear transformation $\varphi : V \to V^{**}$.

II. Group Theory

1. Let G be a nonabelian finite simple group and let H be a subgroup of index p, where p is a prime. Prove that the number of distinct conjugates of H in G is p.

2. Let G be a finite group with a normal subgroup $N \cong S_3$. Show that there is a subgroup H of G such that $G = N \times H$.

3. Let G be a finite group, p a prime, and P a Sylow p-subgroup of G. Let H be a subgroup of G that contains the normalizer $N_G(P)$ of P in G. Show that if g is an element of G such that $g^{-1}Pg \leqslant H$, then g is an element of H.

4. Let G be a group with exactly 31 Sylow 3-subgroups. Prove that there exist Sylow 3-subgroups P and Q satisfying $[P : P \cap Q] = [Q : P \cap Q] = 3$.

5. Show that if G is a finite nilpotent group and m is a positive integer such that m divides the order of G, then G has a subgroup of order m.

III. Ring Theory

1. Let R be a ring with ideals A and B. Let $R/A \times R/B$ be the ring with coordinate-wise addition and multiplication. Show the following.
   (a) The map $R \to R/A \times R/B$ given by $r \mapsto (r + A, r + B)$ is a ring homomorphism.
   (b) The homomorphism in part (a) is surjective if and only if $A + B = R$.

2. Let R be a non-zero ring with identity. Show that every proper ideal of R is contained in a maximal ideal.

3. Let R be a commutative ring with identity and let U be maximal among non-finitely generated ideals of R. Prove U is a prime ideal.

4. Let D be a unique factorization domain such that if p and q are irreducible elements of D, then p and q are associates. Show that if A and B are ideals of D, then either $A \subseteq B$ or $B \subseteq A$.

5. Let D be a principal ideal domain and let P be a non-zero prime ideal. Show that D_P, the localization of D at P, is a principal ideal domain and has a unique irreducible element, up to associates.

IV. Field Theory

1. Let K be a finite degree extension of the field F such that $[K : F]$ is relatively prime to 6. Show that if $u \in K$ then $F(u) = F(u^3)$.

2. If K is an extension of a field F of characteristic $p \neq 0$, then an element u of K is called *purely inseparable* over F if $u^{p^t} \in F$ for some t. Show that the following are equivalent.
   (i) u is purely inseparable over F.
   (ii) u is algebraic over F with minimal polynomial $x^{p^n} - a$ for some $a \in F$ and integer n.
   (iii) u is algebraic over F and its minimal polynomial factors as $(x - u)^m$.

3. Let m be an odd integer and let η_m, η_{2m} be a complex primitive m-th, $2m$-th root of unity, respectively. Show that $\mathbb{Q}(\eta_m) = \mathbb{Q}(\eta_{2m})$.

4. Let $u = \sqrt{2 + \sqrt{2}}$, $v = \sqrt{2 - \sqrt{2}}$, and $E = \mathbb{Q}(u)$, where $\mathbb{Q}$ is the field of rational numbers.
   (a) Find the minimal polynomial $f(x)$ of u over $\mathbb{Q}$.
   (b) Show $v \in E$. Hence conclude that E is a splitting field of $f(x)$ over $\mathbb{Q}$.
   (c) Determine the Galois group of E over $\mathbb{Q}$.

5. Let E and F be finite subfields of a field K. Show that if E and F have the same number of elements, then $E = F$.