QUALIFYING EXAM IN ALGEBRA

August 2022

1. There are 18 problems on the exam. Work and turn in 10 problems, in the following categories.

I.	Linear Algebra	—	1 problem
II.	Group Theory	—	3 problems
III.	Ring Theory	—	2 problems
IV.	Field Theory	—	3 problems
Any of the four areas		—	1 problem

2. Turn in only 10 problems. No credit will be given for extra problems. All problems are weighted equally.

3. Put each problem on a separate sheet of paper, and write only on one side. Put your name on each page.

4. If you feel there is a misprint or error in the statement of a problem, then interpret it in such a way that the problem is not trivial.

I. Linear Algebra

1. Let A be an $n \times n$ Jordan block. Show that any matrix that commutes with A is a polynomial in A.

2. Find all possible Jordan canonical forms of a 5×5 complex matrix with minimal polynomial $(x-2)^2(x-1)$.

3. Let V be a vector space and $T : V \to V$ a linear transformation such that $T^2 = T$. Show that $V = \ker T \oplus \operatorname{Im} T$.

II. Group Theory

1. Let G be any group for which G'/G'' and G''/G''' are cyclic. Prove that $G'' = G'''$.

2. Prove that a finitely generated subgroup of the additive group of the rational numbers must be cyclic.

3. Let P be a finite p-group and let H be a proper subgroup of P. Prove that H is a proper subgroup of its normalizer $N_P(H)$.

4. Let G be a group acting transitively on a set A. Show that if there is an element $a \in A$ such that $G_a = \{1\}$, then $G_b = \{1\}$ for all $b \in A$.

5. Let G be a group of order $3 \cdot 5 \cdot 7 \cdot 13$. Prove that G is not a simple group. [Hint: If a Sylow 7-subgroup is not normal, then some Sylow 13-subgroup will centralize it. Now compute the number of Sylow 13-subgroups.]

III. Ring Theory

1. Let R be a commutative ring with identity and let $x \in R$ be a non-nilpotent element. Prove that there exists a prime ideal P of R such that $x \notin P$.

2. Let R be a commutative ring with identity. Show that if x and y are nilpotent elements of R then $x + y$ is nilpotent and the set of all nilpotent elements is an ideal in R.

3. Let $D = \mathbb{Z}(\sqrt{21}) = \{m + n\sqrt{21} \mid m, n \in \mathbb{Z}\}$ and $F = \mathbb{Q}(\sqrt{21})$, the field of fractions of D. Show the following:
   (a) $x^2 - x - 5$ is irreducible in $D[x]$ but not in $F[x]$.
   (b) D is not a unique factorization domain.

4. Show that if R is an integral domain and $f(x)$ is a unit in the polynomial ring $R[x]$, then $f(x)$ is in R.

5. Let R be a ring with identity such that the identity map is the only ring automorphism of R. Prove that the set N of all nilpotent elements of R is an ideal of R.

IV. Field Theory

1. Let E be a finite dimensional extension of a field F and let G be a group of F-automorphisms of E such that $[E : F] = |G|$. Show that F is the fixed field of G.

2. Show that if K is algebraic over F and $\sigma : K \to K$ is an F-monomorphism, then σ is onto.

3. Show that every field of characteristic 0 is perfect.

4. Let K be a splitting field for $x^5 - 2$ over $\mathbb{Q}$.
   (a) Determine $[K : \mathbb{Q}]$.
   (b) Show that $\mathrm{Gal}(K/\mathbb{Q})$ is non-abelian.
   (c) Find all normal intermediate extensions F and express as $F = \mathbb{Q}(\alpha)$ for appropriate α.

5. Show that any two finite fields of the same order are isomorphic.