QUALIFYING EXAM IN ALGEBRA

August 2023

1. There are 18 problems on the exam. Work and turn in 10 problems, in the following categories.

I.	Linear Algebra	—	1 problem
II.	Group Theory	—	3 problems
III.	Ring Theory	—	2 problems
IV.	Field Theory	—	3 problems
Any of the four areas		—	1 problem

2. Turn in only 10 problems. No credit will be given for extra problems. All problems are weighted equally.

3. Put each problem on a separate sheet of paper, and write only on one side. Put your name on each page.

4. If you feel there is a misprint or error in the statement of a problem, then interpret it in such a way that the problem is not trivial.

I. Linear Algebra

1. Let A and B be complex 2×2 matrices so that $A(AB - BA) = (AB - BA)A$. Prove that the matrix $AB - BA$ is nilpotent.

2. Let A and B be complex 3×3 matrices having the same eigenvectors. Suppose the minimal polynomial of A is $(x-1)^2$ and the characteristic polynomial of B is x^3. Show that the minimal polynomial of B is x^2.

3. A linear transformation $T : V \to W$ is said to be independence preserving if $T(I) \subseteq W$ is linearly independent whenever $I \subseteq V$ is a linearly independent set. Show that T is independence preserving if and only if T is one-to-one.

II. Group Theory

1. Let H be a proper subgroup of the finite group G. Prove that the union of all the conjugates of H is a proper subset of G.

2. Let G be a simple group of order greater than 2 and let $\mathrm{Aut}(G)$ be its automorphism group. Show that the center of $\mathrm{Aut}(G)$ is trivial if and only if G is non-abelian.

3. Show that if G is a subgroup of S_n of index 2, then $G = A_n$.

4. Prove that if G is a simple group containing an element of order 45, then every proper subgroup of G has index at least 14.

5. Prove that a group of order $29 \cdot 30$ has a normal Sylow 29-subgroup.

III. Ring Theory

1. Let R be a commutative ring with 1 and let I be an ideal of R such that for any $a \in I$ the element $1 + a$ is invertible in R. Is it true that all elements of I are nilpotent?

2. A ring R is called simple if $R^2 \neq 0$ and 0 and R are its only ideals. Show that the center of a simple ring is 0 or a field.

3. Let $D = \mathbb{Z}(\sqrt{21}) = \{m + n\sqrt{21} \mid m, n \in \mathbb{Z}\}$ and $F = \mathbb{Q}(\sqrt{21})$, the field of fractions of D. Show the following:
   (a) $x^2 - x - 5$ is irreducible in $D[x]$ but not in $F[x]$.
   (b) D is not a unique factorization domain.

4. Let R be a ring with 1 and suppose that 2 is invertible in R. Suppose that for every x and y in R we have either $xy = yx$, or $xy = -yx$. Prove that R is commutative.

5. Let $F[x, y]$ be the polynomial ring over a field F in two indeterminates x, y. Show that the ideal generated by $\{x, y\}$ is not a principal ideal.

IV. Field Theory

1. Let $\mathbb{Q}$ be the field of rational numbers. Show that the group of automorphisms of $\mathbb{Q}$ is trivial.

2. Let K be a simple algebraic extension of a field F. Show that there are only finitely many intermediate fields between F and K.

3. Show that every finite field is perfect.

4. Determine the Galois group of $x^4 - 3$ over the field $\mathbb{Q}$ of rational numbers.

5. Let F be a finite field. Show that the product of all the non-zero elements of F is -1.