QUALIFYING EXAM IN ALGEBRA

AUGUST 1992

1. Work as many problems as you can. It is to your advantage to demonstrate a broad background.

2. If you feel there is a misprint or error in the statement of the problem, then interpret it in such a way that the problem is not trivial.

Group Theory

1. (a) Find the centralizer in S_5 of $\sigma = (\ 1\ \ 2\ )(\ 3\ \ 4\ \ 5\ )$.

   (b) How many elements of order 6 are there in S_5?

2. Let G be a group and let $Z(G)$ be the center of G. Prove or disprove the following.

   (a) If $G/Z(G)$ is cyclic, then G is abelian.

   (b) If $G/Z(G)$ is abelian, then G is abelian.

   (c) If G is of order p^2, where p is a prime, then G is abelian.

3. Let G be a group. Show that if G has a proper subgroup of finite index, then G has a proper normal subgroup of finite index.

4. Let $Inn(G)$ be the group of inner automorphisms of the group G and let $Aut(G)$ be the full automorphism group.

   (a) Show that $Inn(G) \trianglelefteq Aut(G)$.

   (b) Show that if $Z(G)$ is the center of G, then $Inn(G) \cong G/Z(G)$.

5. Let H be a subgroup of G and suppose there is a normal subgroup N of G satisfying $HN = G$ and $H \cap N = \langle 1 \rangle$. Prove that if two elements of H are conjugate in G, then they are conjugate in H.

6. Show that a group of order $1960 = 2^3 \cdot 5 \cdot 7^2$ cannot be simple.

Ring Theory

1. (a) Show that $x^4 + x^3 + x^2 + x + 1$ is irreducible in $\boldsymbol{Z}_3[x]$.

   (b) Show that $x^4 + 1$ is not irreducible in $\boldsymbol{Z}_3[x]$.

2. Let R be the ring of all 2×2 matrices of the form $\begin{bmatrix} a & b \\ 2b & a \end{bmatrix}$, where $a, b \in \boldsymbol{Z}$. Prove that R is isomorphic to $\boldsymbol{Z}[\sqrt{2}]$.

3. Let D be a principal ideal domain. Prove that every nonzero prime ideal of D is a maximal ideal.

4. Let S be the ring of all bounded, continuous functions $f : \boldsymbol{R} \to \boldsymbol{R}$, where $\boldsymbol{R}$ is the set of real numbers. Let I be the set of functions f in S such that $f(t) \to 0$ as $|t| \to \infty$.

   (a) Show that I is an ideal of S.

   (b) Suppose $x \in S$ is such that there is an $i \in I$ with $ix = x$. Show that $x(t) = 0$ for all sufficiently large $|t|$.

5. Let R be a commutative ring with identity such that not every ideal is a principal ideal.

   (a) Show that there is an ideal I maximal with respect to the property that I is not a principal ideal.

   (b) If I is the ideal of part (a), show that R/I is a principal ideal ring.

6. Let R be a subring of a field F such that for each x in F either $x \in R$ or $x^{-1} \in R$. Prove that if I and J are two ideals of R, then either $I \subseteq J$ or $J \subseteq I$.

Field Theory

1. Let F be a field extension of the rational numbers.
   (a) Show that $\{a + b\sqrt{2} \,|\, a, b \in F\}$ is a field.
   (b) Give necessary and sufficient conditions for $\{a + b\sqrt[3]{2} \,|\, a, b \in F\}$ to be a field.

2. (a) Find the Galois group of $x^3 - 5$ over $\boldsymbol{Q}$ and demonstrate the Galois correspondence between the subgroups of the Galois group and the subfields of the splitting field.
   (b) Find all automorphisms of $\boldsymbol{Q}(\sqrt[3]{5})$. Is there an $f \in \boldsymbol{Q}[x]$ with splitting field $\boldsymbol{Q}(\sqrt[3]{5})$? Explain.

3. Let α be a complex primitive 43^{rd} root of 1. Prove that there is an extension field F of the rational numbers such that $[F(\alpha) : F] = 14$.

4. Let p be a prime. Show that the field of p^m elements is contained in the field of p^n elements if and only if $m|n$.

5. Let $K = F(u)$ be a separable extension of F with $u^m \in F$ for some positive integer m. Show that if the characteristic of F is p and $m = p^t r$, then $u^r \in F$.

6. Let F be a field and let $f(x) \in F[x]$ be an irreducible polynomial of degree 4 with distinct roots α_1, α_2, α_3, and α_4. Let K be a splitting field for f over F and assume $Gal(K/F) \cong S_4$. Find $Gal(K/F(\beta))$, where $\beta = \alpha_1\alpha_2 + \alpha_3\alpha_4$.

Linear Algebra

1. Let V be a finite dimensional vector space over a field F and let $T : V \to V$ be a nilpotent linear transformation. Show that the trace of T is 0.

2. (a) Show that two 3×3 complex matrices are similar if and only if they have the same characteristic and minimal polynomials.

(b) Is the conclusion of part (a) true for larger matrices? Prove or give a counter-example.