

QUALIFYING EXAM IN ALGEBRA

August 1993

1. Work as many problems as you can. It is to your advantage to demonstrate a broad background.
2. If you feel there is a misprint or error in the statement of the problem, then interpret it in such a way that the problem is not trivial.

Linear Algebra

- Show that two 3×3 complex matrices are similar if and only if they have the same characteristic and minimal polynomials.
 - Is the conclusion of part (a) true for larger matrices? Prove or give a counter-example.
- Let V be a finite-dimensional vector space over a field F and let U be a subspace. Show that there is a subspace W of V such that $V = U \oplus W$.

Group Theory

- Give a representative for each conjugacy class of elements of order 6 in S_6 .
 - Find the order of the centralizer in S_6 of each element from part (a).
- Let H be a subgroup of the group G . Show that the following are equivalent:
 - $x^{-1}y^{-1}xy \in H$ for all x and y in G
 - H is a normal subgroup of G and G/H is abelian.
- Consider the property (*) of abelian groups G :

(*) If H is any subgroup of G then there exists a subgroup F of G
such that $G/H \cong F$.

Show that if G is a finitely generated abelian group then G has property (*) if and only if G is finite.
- Let p be a prime and G a nonabelian group of order p^3 . Show that $Z(G)$, the center of G , has order p and that $G/Z(G)$ is isomorphic to $\mathbf{Z}_p \times \mathbf{Z}_p$.
- Let G be a group of order 168 and let P be a Sylow 7-subgroup of G . Show that either P is a normal subgroup of G or else the normalizer of P is a maximal subgroup of G .

Ring Theory

1. Denote the set of invertible elements of the ring \mathbf{Z}_n by U_n .
 - (a) List all the elements of U_{18} .
 - (b) Is U_{18} a cyclic group under multiplication? Justify your answer.
2. Show that the polynomial $f(x) = x^4 + 5x^2 + 3x + 2$ is irreducible over the field of rational numbers.
3. Let R be a finite commutative ring with more than one element that contains no zero-divisors. Show that R is a field. (Note: It is *not* given that R has a unit element.)
4. Show that if R is a commutative ring with 1 in which every ideal is prime, then R is a field. (Hint: For $a \neq 0$ consider the ideals (a) and (a^2) .)

Field Theory

1. Show that if K is a finite extension of a field F then K is an algebraic extension of F .
2. Let K be an extension field of F with $a, b \in K$. Let $[F(a) : F] = m$ and $[F(b) : F] = n$ and assume $(m, n) = 1$. Show that $F(a) \cap F(b) = F$ and $[F(a, b) : F] = mn$.
3.
 - (a) Show that for every prime p and every positive integer n there is an irreducible polynomial of degree n over the field \mathbf{F}_p of p elements.
 - (b) Show that for every positive integer n there is an irreducible polynomial of degree n over the field \mathbf{Q} of rational numbers.
4.
 - (a) Find the Galois group of $x^3 - 2$ over \mathbf{Q} and demonstrate the Galois correspondence between the subgroups of the Galois group and the subfields of the splitting field.
 - (b) Find all automorphisms of $\mathbf{Q}(\sqrt[3]{2})$. Is there an $f(x) \in \mathbf{Q}[x]$ with splitting field $\mathbf{Q}(\sqrt[3]{2})$? Explain.

5. Let $f(x) = x^4 + ax^3 + bx^2 + ax + 1$ be a polynomial with rational coefficients and let K be a splitting field for f over \mathbf{Q} . Show that if α is a root of f then so is $\frac{1}{\alpha}$ and $|\text{Gal}(K/\mathbf{Q})| \leq 8$.