QUALIFYING EXAM IN ALGEBRA

August 1994

1. Work as many problems as you can. It is to your advantage to demonstrate a broad background.

2. If you feel there is a misprint or error in the statement of the problem, then interpret it in such a way that the problem is not trivial.

Linear Algebra

1. Let V be a finite dimensional vector space and $T : V \to V$ a linear transformation.
   (a) Show that T is invertible if and only if the minimal polynomial of T has non-zero constant term.
   (b) Show that if T is invertible, then T^{-1} is expressible as a polynomial in T.

2. A linear transformation $T : V \to W$ is said to be independence preserving if $T(I) \subseteq W$ is linearly independent whenever $I \subseteq V$ is a linearly independent set. Show that T is independence preserving if and only if T is one-to-one.

Group Theory

1. For $i = 1, \ldots, n-1$, let x_i be the transposition $(i \quad i+1)$ in the symmetric group S_n. Show that $S_n = \langle x_1, \ldots, x_{n-1} \rangle$.

2. Let H be a subgroup of G. Show that $C_G(H)$ is a normal subgroup of $N_G(H)$ and $N_G(H)/C_G(H)$ is isomorphic to a subgroup of $Aut(H)$.

3. Show that a group of order $2 \cdot 7 \cdot 13$ must be solvable.

4. Show that if the center of a group G is of index n in G, then every conjugacy class of G has at most n elements.

5. Let A be an abelian, transitive subgroup of S_n. Show that for all $\alpha \in \{1, \ldots, n\}$, the stabilizer A_α of α in A is trivial.

Ring Theory

1. Let R be a commutative ring with 1 and let I and J be ideals of R such that $I + J = R$. Show that $R/(I \cap J) \cong R/I \oplus R/J$.

2. Let R be a commutative ring with 1 such that for every x in R there is an integer $n > 1$ (depending on x) such that $x^n = x$. Show that every prime ideal of R is maximal.

3. Let R be a principal ideal domain. Show that if (a) is a nonzero ideal in R, then there are only finitely many ideals in R containing (a).

4. (a) Show that $x^4 + x^3 + x^2 + x + 1$ is irreducible in $\boldsymbol{Z}_3[x]$.

   (b) Show that $x^4 + 1$ is not irreducible in $\boldsymbol{Z}_3[x]$.

Field Theory

1. Let $F \subseteq E \subseteq K$ be fields with K a normal extension of F. Show that K is a normal extension of E and give an example to show that E need not be a normal extension of F.

2. Let E and F be subfields of a finite field K. Show that if E is isomorphic to F then $E = F$.

3. Identify the Galois group of the polynomial $x^4 - 3$ over the field of rational numbers. (You must prove your answer.)

4. Let $F \subseteq K$ be fields and let $\alpha \in K$ be algebraic over F with minimal polynomial $f(x) \in F[x]$ of degree n. Show that $\{1, \alpha, \ldots, \alpha^{n-1}\}$ is a basis for $F(\alpha)$ over F.

5. Let K be a finite Galois extension of F of characteristic 0. Show that if $\mathrm{Gal}(K/F)$ is a non-trivial 2-group, then there is a quadratic extension of F contained in K.