QUALIFYING EXAM IN ALGEBRA

August 1995

1. There are 17 problems on the exam. Work and turn in 10 problems, in the following categories.

I. Linear Algebra	—	1 problem
II. Group Theory	—	3 problems
III. Ring Theory	—	2 problems
IV. Field Theory	—	3 problems
Any of the four areas	—	1 problem

2. Turn in only 10 problems. No credit will be given for extra problems. All problems are weighted equally.

3. Put each problem on a separate sheet of paper, and write only on one side. Put your name on each page.

4. If you feel there is a misprint or error in the statement of a problem, then interpret it in such a way that the problem is not trivial.

I. Linear Algebra

1. Let A and B be 5×5 complex matrices and suppose that A and B have the same eigenvectors. Show that if the minimal polynomial of A is $(x+1)^2$ and the characteristic polynomial of B is x^5, then $B^3 = 0$.

2. Let $T : V \to W$ be a surjective linear transformation of finite dimensional vector spaces over a field F (acting on the left). Prove that there is a linear transformation $S : W \to V$ such that $T \circ S$ is the identity map on W.

3. Let V be a vector space and let U and W be finite dimensional subspaces of V. Show that $\dim(U + W) = \dim U + \dim W - \dim U \cap W$.

II. Group Theory

1. Prove that a group of order $1995 = 3 \cdot 5 \cdot 7 \cdot 19$ must be solvable.

2. Let G be a group with a normal subgroup N of order 5, such that $G/N \cong S_3$. Show that $|G| = 30$, G has a normal subgroup of order 15, and G has 3 subgroups of order 10 that are not normal.

3. Let G be a finite group. Show that if G has a normal subgroup N of order 3 that is not contained in the center of G, then G has a subgroup of index 2. [Hint: The group G acts on N by conjugation.]

4. Let G be a finite group and let P be a Sylow p-subgroup of G. Prove the following.
   (a) If M is any normal p-subgroup of G then $M \le P$.
   (b) There is a normal p-subgroup N of G that contains all normal p-subgroups of G.

5. Let $G = A \times B$ be a direct product of the subgroups A and B. Suppose H is a subgroup of G that satisfies $HA = G = HB$ and $H \cap A = \langle 1 \rangle = H \cap B$. Prove that A is isomorphic to B.

III. Ring Theory

1. Let R be a ring with ideals A and B. Let $R/A \times R/B$ be the ring with coordinatewise addition and multiplication. Show the following.

   (a) The map $R \to R/A \times R/B$ given by $r \mapsto (r + A, r + B)$ is a ring homomorphism.

   (b) The homomorphism in part (a) is surjective if and only if $A + B = R$.

2. Let R be a non-zero commutative ring with 1.

   (a) Let S be a multiplicative subset of R not containing 0 and let P be maximal in the set of ideals of R not intersecting S. Show that P is a prime ideal.

   (b) Show that the set of nilpotent elements of R is the intersection of all prime ideals.

3. Let R be a commutative ring with 1. An ideal I of R is called a *primary* ideal if $I \neq R$ and for all $x, y \in R$ with $xy \in I$, either $x \in I$ or $y^n \in I$ for some integer $n \geq 1$.

   (a) Show that an ideal I of R is primary if and only if $R/I \neq \{0\}$ and every zero-divisor in R/I is nilpotent.

   (b) Show that if I is a primary ideal of R then the radical $\mathrm{Rad}(I)$ of I is a prime ideal. (Recall that $\mathrm{Rad}(I) = \{x \in R \,|\, x^n \in I \text{ for some } n\}$.)

4. A *local* ring is a commutative ring with 1 that has a unique maximal ideal. Show that a ring R is local if and only if the set of non-units in R is an ideal.

IV. Field Theory

1. Find the minimal polynomial of $\alpha = \sqrt{3+\sqrt{7}}$ over the field $\boldsymbol{Q}$ of rational numbers, and *prove* it is the minimal polynomial.

2. Let α be algebraic over F with minimal polynomial $f(x) \in F[x]$ and let $F \subseteq E \subseteq F(\alpha)$. Show that if $E \neq F$, then $f(x)$ is *not* irreducible over E.

3. Let η be a complex primitive 7-th root of unity and let $K = \boldsymbol{Q}(\eta)$. Find $\mathrm{Gal}(K/\boldsymbol{Q})$ and express each intermediate field F between K and $\boldsymbol{Q}$ as $F = \boldsymbol{Q}(\beta)$ for some $\beta \in K$.

4. Suppose K is a Galois extension of a field F, but that there exists a field L with $F \subseteq L \subseteq K$ such that L is not a Galois extension of F. Show that the Galois group of K over F is not abelian.

5. A field F is called *perfect* if every element of an algebraic closure of F is separable over F. Let F be a field of characteristic p. Show that the following are equivalent.
   (i) The field F is perfect.
   (ii) For every $\epsilon \in F$ there exists a $\delta \in F$ such that $\delta^p = \epsilon$.