QUALIFYING EXAM IN ALGEBRA

August 1996

1. There are 17 problems on the exam. Work and turn in 10 problems, in the following categories.

I. Linear Algebra	—	1 problem
II. Group Theory	—	3 problems
III. Ring Theory	—	2 problems
IV. Field Theory	—	3 problems
Any of the four areas	—	1 problem

2. Turn in only 10 problems. No credit will be given for extra problems. All problems are weighted equally.

3. Put each problem on a separate sheet of paper, and write only on one side. Put your name on each page.

4. If you feel there is a misprint or error in the statement of a problem, then interpret it in such a way that the problem is not trivial.

I. Linear Algebra

1. Find all possible minimal polynomials for a complex matrix with characteristic polynomial $(x-3)^4(x-5)^2$ and for each such minimal polynomial, determine all possible Jordan canonical forms of the matrix.

2. Let V be the vector space of $n \times n$ matrices over the field $\boldsymbol{R}$ of real numbers. Let U be the subspace of V consisting of symmetric matrices and W the subspace of V consisting of skew-symmetric matrices. Show that $V = U \oplus W$.

3. Prove the Cayley-Hamilton Theorem: every square matrix is a zero of its characteristic polynomial.

II. Group Theory

1. Show that if G is a non-abelian simple group and G is a subgroup of S_n, then G is a subgroup of A_n.

2. Let $N \trianglelefteq G$ such that every subgroup of N is normal in G and $C_G(N) \subseteq N$. Prove that G/N is abelian.

3. Let G be a group of order $105 = 3 \cdot 5 \cdot 7$. Prove that a Sylow 7-subgroup of G is normal.

4. Let G be a finite group and let p be the smallest prime dividing the order of G. Show that if H is a subgroup of G of index p, then H is a normal subgroup of G.

5. We say that a group X is *involved* in a group G if X is isomorphic to H/K for some subgroups K, H of G with $K \trianglelefteq H$. Prove that if X is solvable and X is involved in the finite group G, then X is involved in a solvable subgroup of G.

III. Ring Theory

1. Let R be a commutative ring with identity. Show that if x and y are nilpotent elements of R then $x + y$ is nilpotent and the set of all nilpotent elements is an ideal in R.

2. Let R be a commutative ring with 1 and D a multiplicative subset of R containing 1. Let J be an ideal in the ring of fractions $D^{-1}R$ and let

$$I = \{\, a \in R \mid \frac{a}{d} \in J \text{ for some } d \in D \,\}.$$

Show that I is an ideal of R.

3. Let $D = \{a + b\sqrt{17} \,|\, a, b \in \boldsymbol{Z}\}$ and let $F = \boldsymbol{Q}(\sqrt{17})$ be the field of fractions of D.
   (a) Show that $x^2 + x - 4$ is irreducible over D but not over F.
   (b) Show that D is not a unique factorization domain.

4. Let R be a commutative ring with identity such that not every ideal is a principal ideal.
   (a) Show that there is an ideal I maximal with respect to the property that I is not a principal ideal.
   (b) Show that the ideal I found in part (a) is a prime ideal.

IV. Field Theory

1. Let K be a field extension of F of degree n and let $f(x) \in F[x]$ be an irreducible polynomial of degree $m > 1$. Show that if m is relatively prime to n, then f has no root in K.

2. Let K be an extension field of F and let α be an element of K. Show that the following are equivalent:
   (i) α is algebraic over F,
   (ii) $F(\alpha)$ is a finite dimensional extension of F,
   (iii) α is contained in a finite dimensional extension of F.

3. Let F be a field and let $f(x) \in F[x]$ have splitting field K. Show that if the degree of f is a prime p and $[K : F] = tp$ for some integer t, then
   (a) $f(x)$ is irreducible over F and
   (b) if $t > 1$ then K is a separable extension of F.

4. Let K be a Galois extension of $\boldsymbol{Q}$ with $\mathrm{Gal}(K/\boldsymbol{Q}) \cong S_5$. Show that K is the splitting field of a polynomial of degree 5 over $\boldsymbol{Q}$.

5. Let F be an extension of $\boldsymbol{Z}_p$ of degree n. Show that F is a Galois extension and $\mathrm{Gal}(F/\boldsymbol{Z}_p)$ is cyclic of order n.