

QUALIFYING EXAM IN ALGEBRA

August 1997

1. There are 18 problems on the exam. Work and turn in 10 problems, in the following categories.

I. Linear Algebra	—	1 problem
II. Group Theory	—	3 problems
III. Ring Theory	—	2 problems
IV. Field Theory	—	3 problems
Any of the four areas	—	1 problem

2. Turn in only 10 problems. No credit will be given for extra problems. All problems are weighted equally.
3. Put each problem on a separate sheet of paper, and write only on one side. Put your name on each page.
4. If you feel there is a misprint or error in the statement of a problem, then interpret it in such a way that the problem is not trivial.

I. Linear Algebra

1. Let A be an $n \times n$ nilpotent matrix such that $A^{n-1} \neq 0$. Show that A has exactly one Jordan block.
2. (a) Give an example of a complex 2×2 matrix which does not have a square root.
(b) Show that every complex non-singular $n \times n$ matrix has a square root. [Hint: Show first that a Jordan block with non-zero eigenvalue has a square root.]
3. Let $T : V \rightarrow W$ be a linear transformation of vector spaces over a field F .
(a) Show that T is injective if and only if $\{T(v_1), \dots, T(v_n)\}$ is linearly independent in W whenever $\{v_1, \dots, v_n\}$ is linearly independent in V .
(b) Show that T is surjective if and only if $\{T(x) \mid x \in X\}$ is a spanning set for W whenever X is a spanning set for V .

II. Group Theory

1. Let $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \in S_6$.
(a) Determine the size of the conjugacy class of σ and the order of the centralizer of σ in S_6 .
(b) Determine if $C_{S_6}(\sigma)$ is abelian or non-abelian. Prove your answer.
2. Let $a \in G$ be fixed, where G is a group. Prove that a commutes with each of its conjugates in G if and only if a belongs to an abelian normal subgroup of G .
3. Let H be a subgroup of G of index 2 and let g be an element of H . Show that if $C_G(g) \subseteq H$ then the conjugacy class of g in G splits into 2 conjugacy classes in H , and if $C_G(g) \not\subseteq H$, then the class of g in G remains the class of g in H .
4. Let p and q be distinct primes. Show that a group of order p^2q has a normal Sylow p -subgroup or a normal Sylow q -subgroup.
5. Let G be a group acting transitively on the set Ω . Show that if $\alpha \neq \beta$ are elements of Ω , then $G_\alpha G_\beta$ is a proper subset of G .

III. Ring Theory

1. Give an example of each of the following. (You need not verify the example.)
 - (a) An infinite non-commutative ring with non-zero characteristic.
 - (b) An integral domain which is not a unique factorization domain.
 - (c) A commutative ring with a sequence $\{P_n\}_{n=1}^{\infty}$ of prime ideals such that P_n is properly contained in P_{n+1} for all n .
 - (d) A commutative ring that has exactly one maximal ideal and is not a field.
 - (e) A non-commutative ring with exactly two maximal ideals.
2. Let D be an integral domain and let c be an irreducible element in D . Show that the ideal (x, c) generated by x and c in the polynomial ring $D[x]$ is not a principal ideal.
3. The *Jacobson Radical* $J(R)$ of a ring R is defined to be the intersection of all maximal ideals of R . Let R be a commutative ring with 1 and let $x \in R$. Show that $x \in J(R)$ if and only if $1 - xy$ is a unit for all y in R .
4.
 - (a) How many units does the ring $\mathbf{Z}/60\mathbf{Z}$ have? Explain your answer.
 - (b) How many ideals does the ring $\mathbf{Z}/60\mathbf{Z}$ have? Explain your answer.
5. Let R be a commutative ring and P a prime ideal of R . Show that there is a prime ideal $P_0 \subseteq P$ which does not properly contain any prime ideal.

IV. Field Theory

1. Let E be an extension field of F and $A = \{e \in E \mid e \text{ is algebraic over } F\}$.
 - (a) Show that A is a subfield of E containing F .
 - (b) Show that if $\sigma : E \rightarrow E$ is a one-to-one F -homomorphism, then $\sigma(A) = A$.
2. An extension K of a field F of characteristic $p \neq 0$, is a *purely inseparable* extension if for each $u \in K$ there is an integer t such that $u^{p^t} \in F$. Show that every purely inseparable field extension is a normal extension.
3. Let F be a field of characteristic p . Fix $c \in F$ and let $f(x) = x^p - x + c \in F[x]$. Prove that if α is a root of $f(x)$ in some extension field, then so is $\alpha + 1$. Use this to prove that if K is the splitting field of $f(x)$ over F , then either $K = F$ and $f(x)$ splits completely over F , or $[K : F] = p$ and $f(x)$ is irreducible over F . (Hint: Use Galois groups.)
4. Let η be a complex primitive 5th root of unity and let $K = \mathbf{Q}(\eta)$, where \mathbf{Q} is the field of rational numbers. Show that there is a unique extension F of degree 2 of \mathbf{Q} contained in K and find $q \in \mathbf{Q}$ such that $F = \mathbf{Q}(\sqrt{q})$.
5. Suppose $K = F(\alpha)$ is a proper Galois extension of F and assume there exists an element σ of $\text{Gal}(K/F)$ satisfying $\sigma(\alpha) = \alpha^{-1}$. Show that $[K : F]$ is even and that $[F(\alpha + \alpha^{-1}) : F] = \frac{1}{2}[K : F]$.