QUALIFYING EXAM IN ALGEBRA

August 1998

1. There are 18 problems on the exam. Work and turn in 10 problems, in the following categories.

I.	Linear Algebra	—	1 problem
II.	Group Theory	—	3 problems
III.	Ring Theory	—	2 problems
IV.	Field Theory	—	3 problems
Any of the four areas		—	1 problem

2. Turn in only 10 problems. No credit will be given for extra problems. All problems are weighted equally.

3. Put each problem on a separate sheet of paper, and write only on one side. Put your name on each page.

4. If you feel there is a misprint or error in the statement of a problem, then interpret it in such a way that the problem is not trivial.

I. Linear Algebra

1. Let $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix}$.

   Find the characteristic and minimal polynomials of A and determine the Jordan canonical form of A.

2. Let V be a vector space over a field F. A linear transformation $T : V \to V$ is said to be *idempotent* if $T^2 = T$. Prove that if T is idempotent then $V = V_0 \oplus V_1$, where $T(v_0) = 0$ for all $v_0 \in V_0$ and $T(v_1) = v_1$ for all $v_1 \in V_1$.

3. Let V be a finite dimensional vector space and let W be a subspace. Show that $\dim V/W = \dim V - \dim W$.

II. Group Theory

1. Show that if $\sigma \in S_n$ is an $(n-1)$-cycle, where $n \geq 3$, then $C_{S_n}(\sigma) = \langle \sigma \rangle$.

2. Let N be a normal subgroup of G. Show that if $N \cap G' = \langle 1 \rangle$, then N is contained in the center of G.

3. Let G be a group acting on the set S and let H be a subgroup of G acting transitively on S. Show that if $t \in S$ then $G = G_t H$, where G_t is the stabilizer of t in G.

4. Show that a group of order $1998 = 2 \cdot 3^3 \cdot 37$ must be solvable.

5. A subgroup H of a group G is subnormal if there exists a chain $H = H_0 \leq H_1 \leq \cdots \leq H_k = G$ such that H_i is a normal subgroup of H_{i+1} for every i. Prove that if P is a Sylow p-subgroup of a finite group G then P is a subnormal in G if and only if P is normal in G.

III. Ring Theory

1. Let R be a commutative ring with 1 and let I and J be ideals of R such that $I+J=R$. Show that $R/(I\cap J)\cong R/I\oplus R/J$.

2. Let R be a non-zero ring with identity. Show that every proper ideal of R is contained in a maximal ideal.

3. Let R be a non-zero commutative ring with 1. Show that if I is an ideal of R such that $1+a$ is a unit in R for all $a\in I$, then I is contained in every maximal ideal of R.

4. Let D be a unique factorization domain and F its field of fractions. Prove that if d is an irreducible element in D, then there is no $x\in F$ such that $x^2=d$.

5. Let R be an integral domain, S a multiplicative set, and let $S^{-1}R=\{\frac{r}{s}\,|\,r\in R,\, s\in S\}$ (contained in the field of fractions of R). Show that if P is a prime ideal of R then, $S^{-1}P$ is either a prime ideal of $S^{-1}R$ or else equals $S^{-1}R$.

IV. Field Theory

1. Let K be a finite degree extension of the field F such that $[K:F]$ is relatively prime to 6. Show that if $u\in K$ then $F(u)=F(u^3)$.

2. Let F be a field and $f(x)\in F[x]$ an irreducible polynomial. Prove that there is a prime p, an integer $a\geq 0$ and a separable polynomial $g(x)\in F[x]$ such that $f(x)=g(x^{p^a})$.

3. Show that the Galois group of x^3-7 over $\boldsymbol{Q}$ is S_3 and demonstrate the Galois correspondence between the subgroups of S_3 and the subfields of the splitting field. Which subfields are normal over $\boldsymbol{Q}$?

4. Let $f(x)\in\boldsymbol{Q}[x]$ be an irreducible polynomial of degree 4 with exactly 2 real roots. Show that the Galois group of f is either S_4 or the dihedral group of order 8.

5. Let $\boldsymbol{F}_q$ be the field of q elements and let $f(x)$ be a polynomial in $\boldsymbol{F}_q[x]$. Show that if α is a root of $f(x)$ in some extension of $\boldsymbol{F}_q$, then α^q is also a root of $f(x)$.