QUALIFYING EXAM IN ALGEBRA

January 2005

1. There are 18 problems on the exam. Work and turn in 10 problems, in the following categories.

   I. Linear Algebra — 1 problem
   II. Group Theory — 3 problems
   III. Ring Theory — 2 problems
   IV. Field Theory — 3 problems
   Any of the four areas — 1 problem

2. Turn in only 10 problems. No credit will be given for extra problems. All problems are weighted equally.

3. Put each problem on a separate sheet of paper, and write only on one side. Put your name on each page.

4. If you feel there is a misprint or error in the statement of a problem, then interpret it in such a way that the problem is not trivial.

I. Linear Algebra

1. Let U, V, and W be vector spaces over a field F and let $S : U \to V$ and $T : V \to W$ be linear transformations such that $T \circ S = \mathbf{0}$, the zero map. Show that

$$\dim(W/\mathrm{Im}\ T) - \dim(\ker T/\mathrm{Im}\ S) + \dim \ker S = \dim W - \dim V + \dim U.$$

2. Let $G = GL_n(\mathbb{C})$ be the multiplicative group of invertible $n \times n$ matrices with complex entries and let g be an element of G of finite order. Show that g is diagonalizable.

3. Let A be a 5×5 matrix with complex entries such that $A^3 = 0$. Find all possible Jordan Canonical Forms for A.

II. Group Theory

1. Prove that the symmetric group S_n is a maximal subgroup of S_{n+1}.
   [Hint: Show that if $g \in S_{n+1} - S_n$, then $S_{n+1} = S_n \cup S_n g S_n$.]

2. Let G be a finite group and let M be a maximal subgroup of G. Show that if M is a normal subgroup of G, then $|G : M|$ is prime.

3. Let p be a prime and let G be a non-abelian group of order p^3.
   (a) Show that the center $Z(G)$ of G and the commutator subgroup of G are equal and of order p.
   (b) Show that $G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

4. Let G be a group acting on the set S and let H be a subgroup of G acting transitively on S. Show that if $t \in S$, then $G = G_t H$, where G_t is the stabilizer of t in G.

5. Show that if G is a group of order $392 = 2^3 \cdot 7^2$, then G has a normal subgroup of order 7 or a normal subgroup of order 49.

III. Ring Theory

1. Let R be a commutative ring with 1. Show that the sum of any two principal ideals of R is principal if and only if every finitely generated ideal of R is principal.

2. Let R_1 and R_2 be commutative rings with identities and let $R = R_1 \times R_2$. Show that every ideal I of R is of the form $I = I_1 \times I_2$ with I_i an ideal of R_i for $i = 1, 2$.

3. Find all values of a in $\mathbb{Z}_5$ such that the quotient ring

$$\mathbb{Z}_5[x]/(x^3 + 2x^2 + ax + 3)$$

   is a field. Justify your answer.

4. Let $D = \mathbb{Z}(\sqrt{21}) = \{\, m + n\sqrt{21} \mid m, n \in \mathbb{Z} \,\}$ and $F = \mathbb{Q}(\sqrt{21})$, the field of fractions of D. Show the following:
   (a) $x^2 - x - 5$ is irreducible in $D[x]$ but not in $F[x]$.
   (b) D is not a unique factorization domain.

5. Let R be a non-zero commutative ring with 1 and S a multiplicative subset of R not containing 0. Show that if P is maximal in the set of ideals of R not intersecting S, then P is a prime ideal.

IV. Field Theory

1. Let α be algebraic over the field F with minimal polynomial $f(x) \in F[x]$ and let $K = F[\alpha]$. Show that if $\sigma : F \to L$ is a field monomorphism and $\beta \in L$ is a root of $f^\sigma(x) \in L[x]$, then σ has a unique extension $\hat{\sigma} : K \to L$ satisfying $\hat{\sigma}(\alpha) = \beta$.

2. Let K be a simple algebraic extension of a field F. Show that there are only finitely many intermediate fields between F and K.

3. Let η be a complex primitive 11th root of unity and let $K = \mathbb{Q}(\eta)$, where $\mathbb{Q}$ is the field of rational numbers. Show that there is a unique extension F of degree 2 of $\mathbb{Q}$ contained in K and find $q \in \mathbb{Q}$ such that $F = \mathbb{Q}(\sqrt{q})$.

4. Let $\alpha = \sqrt{5 + 2\sqrt{5}}$. Show that $\mathbb{Q}(\alpha)$ is a cyclic Galois extension of $\mathbb{Q}$ of degree 4. Find all fields F with $\mathbb{Q} \subseteq F \subseteq \mathbb{Q}(\alpha)$.
   [Hint: Show that $f(x) = x^4 - 10x^2 + 5$ is the minimal polynomial of α over $\mathbb{Q}$ and that the roots of f are $\pm\alpha$, $\pm\frac{\sqrt{5}}{\alpha}$.]

5. (a) Define *Galois Extension.*

   (b) Show that every finite extension of a finite field is a Galois extension.