QUALIFYING EXAM IN ALGEBRA

January 2006

1. There are 18 problems on the exam. Work and turn in 10 problems, in the following categories.

I.	Linear Algebra	—	1 problem
II.	Group Theory	—	3 problems
III.	Ring Theory	—	2 problems
IV.	Field Theory	—	3 problems
Any of the four areas		—	1 problem

2. Turn in only 10 problems. No credit will be given for extra problems. All problems are weighted equally.

3. Put each problem on a separate sheet of paper, and write only on one side. Put your name on each page.

4. If you feel there is a misprint or error in the statement of a problem, then interpret it in such a way that the problem is not trivial.

I. Linear Algebra

1. Let $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

   (a) Find the characteristic polynomial of A.

   (b) Find the minimal polynomial of A.

   (c) Find the eigenvalues of A.

   (d) Find the dimensions of all eigenspaces of A.

   (e) Find the Jordan canonical form of A.

2. (a) Prove that a 2×2 scalar matrix A over a field F has a square root (i.e., a matrix B satisfying $B^2 = A$).

   (b) Prove that a real symmetric matrix having the property that every negative eigenvalue occurs with even multiplicity has a square root. [Hint: Use (a).]

3. Let A, B, and C be subspaces of the nonzero vector space V satisfying

$$V = A \oplus B = B \oplus C = A \oplus C.$$

   Show that there exists a 2-dimensional subspace $W \subseteq V$ such that each of $W \cap A$, $W \cap B$, and $W \cap C$ has dimension 1.

II. Group Theory

1. Show that if H is a cyclic normal subgroup of a finite group G, then every subgroup of H is a normal subgroup of G.

2. Let G be a finite group, H a subgroup of G of index 2, and $x \in H$. Denote by $c\ell_G(x)$ the conjugacy class of x in G and by $c\ell_H(x)$ the conjugacy class of x in H.
   (a) Show that if $C_G(x) \leqslant H$, then $|c\ell_H(x)| = \frac{1}{2}|c\ell_G(x)|$.

   (b) Show that if $C_G(x)$ is not contained in H, then $|c\ell_H(x)| = |c\ell_G(x)|$.

   [Hint: Consider centralizer orders.]

3. Let $n > 1$ be a fixed integer. Prove that there are only finitely many simple groups (up to isomorphism) containing a proper subgroup of index less than or equal to n.

4. Show that a group of order $160 = 2^5 \cdot 5$ must contain a nontrivial normal 2-subgroup.

5. Let G be a solvable group and N a nontrivial normal subgroup of G. Show that there is a nontrivial abelian subgroup A of N with A normal in G.

III. Ring Theory

In the following problems, all rings are nonzero rings with 1 and all modules are unital.

1. Let R be an integral domain. Construct the field of fractions F of R by defining the set F and the two binary operations, and show that the two operations are well-defined. Show that F has a multiplicative identity element and that every nonzero element of F has a multiplicative inverse.

2. Let R be a commutative ring such that not every ideal is a principal ideal.
   (a) Show that there is an ideal I maximal with respect to the property that I is not a principal ideal.
   (b) If I is the ideal of part (a), show that R/I is a principal ideal ring.

3. Let D be an integral domain.
   (a) For $a, b \in D$ define a *greatest common divisor* of a and b.
   (b) For $x \in D$ denote $(x) = \{dx \mid d \in D\}$. Prove that if $(a)+(b) = (d)$, then d is a greatest common divisor of a and b.

4. Let R be a commutative ring.
   (a) Prove that (x) is a prime ideal in $R[x]$ if and only if R is an integral domain.
   (b) Prove that (x) is a maximal ideal in $R[x]$ if and only if R is a field.

5. Let M be an R-module that is generated by finitely many simple submodules. Prove that M is a direct sum of finitely many simple R-modules.

IV. Field Theory

1. Let $f(x)$ and $g(x)$ be irreducible polynomials in $F[x]$ of degrees m and n, respectively, where $(m,n)=1$. Show that if α is a root of $f(x)$ in some field extension of F, then $g(x)$ is irreducible in $F(\alpha)[x]$.

2. Let K be an algebraic extension of F. Show that the following are equivalent.
   (i) Each irreducible polynomial in $F[x]$ with one root in K has all its roots in K.
   (ii) Every F-isomorphism of K into a fixed algebraic closure is an F-automorphism.

3. Let $f(x) = x^4 + 4x^2 + 2$ and let K be the splitting field of f over $\mathbb{Q}$. Show that the Galois group of K over $\mathbb{Q}$ is cyclic of order 4.

4. Let $(m,n)=1$ and let η_j denote a complex primitive j-th root of unity for any positive integer j. Show that $\mathbb{Q}(\eta_{mn}) = \langle \mathbb{Q}(\eta_m), \mathbb{Q}(\eta_n) \rangle$ and $\mathbb{Q}(\eta_m) \cap \mathbb{Q}(\eta_n) = \mathbb{Q}$.

5. Show that every algebraic extension of a finite field is separable.