QUALIFYING EXAM IN ALGEBRA

January 2006

1. There are 18 problems on the exam. Work and turn in 10 problems, in
the following categories.

   I. Linear Algebra — 1 problem
   II. Group Theory — 3 problems
   III. Ring Theory — 2 problems
   IV. Field Theory — 3 problems
   Any of the four areas — 1 problem

2. Turn in only 10 problems. No credit will be given for extra problems.
   All problems are weighted equally.

3. Put each problem on a separate sheet of paper, and write only on one
   side. Put your name on each page.

4. If you feel there is a misprint or error in the statement of a problem,
   then interpret it in such a way that the problem is not trivial.
I. Linear Algebra

1. Let \( A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \).

   (a) Find the characteristic polynomial of \( A \).
   
   (b) Find the minimal polynomial of \( A \).
   
   (c) Find the eigenvalues of \( A \).
   
   (d) Find the dimensions of all eigenspaces of \( A \).
   
   (e) Find the Jordan canonical form of \( A \).

2. (a) Prove that a \( 2 \times 2 \) scalar matrix \( A \) over a field \( F \) has a square root (i.e., a matrix \( B \) satisfying \( B^2 = A \)).

   (b) Prove that a real symmetric matrix having the property that every negative eigenvalue occurs with even multiplicity has a square root. [Hint: Use (a).]

3. Let \( A, B, \) and \( C \) be subspaces of the nonzero vector space \( V \) satisfying

   \[ V = A \oplus B = B \oplus C = A \oplus C. \]

   Show that there exists a 2-dimensional subspace \( W \subseteq V \) such that each of \( W \cap A, W \cap B, \) and \( W \cap C \) has dimension 1.
II. Group Theory

1. Show that if \( H \) is a cyclic normal subgroup of a finite group \( G \), then every subgroup of \( H \) is a normal subgroup of \( G \).

2. Let \( G \) be a finite group, \( H \) a subgroup of \( G \) of index 2, and \( x \in H \).
   Denote by \( \text{cl}_G(x) \) the conjugacy class of \( x \) in \( G \) and by \( \text{cl}_H(x) \) the conjugacy class of \( x \) in \( H \).
   
   (a) Show that if \( C_G(x) \subseteq H \), then \( |\text{cl}_H(x)| = \frac{1}{2} |\text{cl}_G(x)| \).
   
   (b) Show that if \( C_G(x) \) is not contained in \( H \), then \( |\text{cl}_H(x)| = |\text{cl}_G(x)| \).
   
   [Hint: Consider centralizer orders.]

3. Let \( n > 1 \) be a fixed integer. Prove that there are only finitely many simple groups (up to isomorphism) containing a proper subgroup of index less than or equal to \( n \).

4. Show that a group of order \( 160 = 2^5 \cdot 5 \) must contain a nontrivial normal 2-subgroup.

5. Let \( G \) be a solvable group and \( N \) a nontrivial normal subgroup of \( G \).
   Show that there is a nontrivial abelian subgroup \( A \) of \( N \) with \( A \) normal in \( G \).
III. Ring Theory

In the following problems, all rings are nonzero rings with 1 and all modules are unital.

1. Let $R$ be an integral domain. Construct the field of fractions $F$ of $R$ by defining the set $F$ and the two binary operations, and show that the two operations are well-defined. Show that $F$ has a multiplicative identity element and that every nonzero element of $F$ has a multiplicative inverse.

2. Let $R$ be a commutative ring such that not every ideal is a principal ideal.
   (a) Show that there is an ideal $I$ maximal with respect to the property that $I$ is not a principal ideal.
   (b) If $I$ is the ideal of part (a), show that $R/I$ is a principal ideal ring.

3. Let $D$ be an integral domain.
   (a) For $a, b \in D$ define a greatest common divisor of $a$ and $b$.
   (b) For $x \in D$ denote $(x) = \{dx | d \in D\}$. Prove that if $(a) + (b) = (d)$, then $d$ is a greatest common divisor of $a$ and $b$.

4. Let $R$ be a commutative ring.
   (a) Prove that $(x)$ is a prime ideal in $R[x]$ if and only if $R$ is an integral domain.
   (b) Prove that $(x)$ is a maximal ideal in $R[x]$ if and only if $R$ is a field.

5. Let $M$ be an $R$-module that is generated by finitely many simple submodules. Prove that $M$ is a direct sum of finitely many simple $R$-modules.
IV. Field Theory

1. Let $f(x)$ and $g(x)$ be irreducible polynomials in $F[x]$ of degrees $m$ and $n$, respectively, where $(m, n) = 1$. Show that if $\alpha$ is a root of $f(x)$ in some field extension of $F$, then $g(x)$ is irreducible in $F(\alpha)[x]$.

2. Let $K$ be an algebraic extension of $F$. Show that the following are equivalent.
   
   (i) Each irreducible polynomial in $F[x]$ with one root in $K$ has all its roots in $K$.
   
   (ii) Every $F$-isomorphism of $K$ into a fixed algebraic closure is an $F$-automorphism.

3. Let $f(x) = x^4 + 4x^2 + 2$ and let $K$ be the splitting field of $f$ over $\mathbb{Q}$. Show that the Galois group of $K$ over $\mathbb{Q}$ is cyclic of order 4.

4. Let $(m, n) = 1$ and let $\eta_j$ denote a complex primitive $j$-th root of unity for any positive integer $j$. Show that $\mathbb{Q}(\eta_{mn}) = \langle \mathbb{Q}(\eta_m), \mathbb{Q}(\eta_n) \rangle$ and $\mathbb{Q}(\eta_m) \cap \mathbb{Q}(\eta_n) = \mathbb{Q}$.

5. Show that every algebraic extension of a finite field is separable.