QUALIFYING EXAM IN ALGEBRA

January 2018

1. There are 18 problems on the exam. Work and turn in 10 problems, in the following categories.

I.	Linear Algebra	—	1 problem
II.	Group Theory	—	3 problems
III.	Ring Theory	—	2 problems
IV.	Field Theory	—	3 problems
Any of the four areas		—	1 problem

2. Turn in only 10 problems. No credit will be given for extra problems. All problems are weighted equally.

3. Put each problem on a separate sheet of paper, and write only on one side. Put your name on each page.

4. If you feel there is a misprint or error in the statement of a problem, then interpret it in such a way that the problem is not trivial.

I. Linear Algebra

1. Prove that an $n \times n$ complex matrix A is diagonalizable if and only if the minimal polynomial of A has distinct roots.

2. Find the characteristic polynomial of the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & -c_0 \\ 1 & 0 & 0 & \dots & 0 & -c_1 \\ 0 & 1 & 0 & \dots & 0 & -c_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \dots & 0 & -c_{n-2} \\ 0 & 0 & 0 & \dots & 1 & -c_{n-1} \end{bmatrix}.$$

3. Let V be a finite dimensional vector space over the field F. Let V^* be the dual space of V (that is, V^* is the vector space of linear transformations $T : V \to F$). Show that $V \cong V^*$.

II. Group Theory

1. Let G be a group, and let $g \in G$ be an element of order greater than 2 (possibly infinite) such that the conjugacy class of g in G has an odd number of elements. Prove that g is not conjugate to g^{-1}.

2. Let N be a normal subgroup of the group G. Show that if $N \cap G' = 1$, then N is contained in the center of G.

3. Let G be a group with a normal subgroup N of order 5, such that $G/N \cong S_3$. Show that $|G| = 30$, G has a normal subgroup of order 15, and G has 3 subgroups of order 10 that are not normal.

4. Let n be a positive integer and let $A = Z^n$. Prove that if B is any subgroup of A that is generated by fewer than n elements, then the index $|A : B|$ is infinite.

5. Show that a simple group of order 168 must be isomorphic to a subgroup of the alternating group A_8.

III. Ring Theory

1. Let R be the ring of all 2×2 matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ where a and b are real numbers. Prove that R is isomorphic to $\mathbb{C}$, the field of complex numbers.

2. Let R be a commutative ring with identity. Suppose R contains an idempotent element a other than 0 or 1. Show that every prime ideal in R contains an idempotent element other than 0 or 1. (An element $a \in R$ is an idempotent if $a^2 = a$.)

3. Show that if p is a prime such that $p \equiv 1 (\text{mod } 4)$, then $\mathbb{Z}[\sqrt{p}]$ is not a unique factorization domain.

4. Let R be a ring with identity such that the identity map is the only ring automorphism of R. Prove that the set N of all nilpotent elements of R is an ideal of R.

5. Let R be a commutative ring with identity. Prove that any non-empty set of prime ideals of R contains maximal and minimal elements.

IV. Field Theory

1. Show that $p(x) = x^3 + x - 6$ is irreducible over $\mathbb{Q}[\sqrt{-1}]$.

2. Let $f(x)$ and $g(x)$ be irreducible polynomials in $F[x]$ of degrees m and n, respectively, where $(m, n) = 1$. Show that if a is a root of $f(x)$ in some field extension of F, then $g(x)$ is irreducible in $F(a)[x]$.

3. Let K be the splitting field of $x^2 + 2$ over $\mathbb{Q}$. Prove or disprove that $i = \sqrt{-1}$ is an element of K.

4. Show that every finite field is perfect. A field F is called perfect if every element of an algebraic closure of F is separable over F.

5. Determine the Galois group of $x^3 + 3x^2 - 1$ over $\mathbb{Q}$.