QUALIFYING EXAM IN ALGEBRA

January 2018

1. There are 18 problems on the exam. Work and turn in 10 problems, in the following categories.

   I. Linear Algebra — 1 problem
   II. Group Theory — 3 problems
   III. Ring Theory — 2 problems
   IV. Field Theory — 3 problems
   Any of the four areas — 1 problem

2. Turn in only 10 problems. No credit will be given for extra problems. All problems are weighted equally.

3. Put each problem on a separate sheet of paper, and write only on one side. Put your name on each page.

4. If you feel there is a misprint or error in the statement of a problem, then interpret it in such a way that the problem is not trivial.
I. Linear Algebra

1. Prove that an $n \times n$ complex matrix $A$ is diagonalizable if and only if the minimal polynomial of $A$ has distinct roots.

2. Find the characteristic polynomial of the matrix

\[
A = \begin{bmatrix}
0 & 0 & 0 & \ldots & 0 & -c_0 \\
1 & 0 & 0 & \ldots & 0 & -c_1 \\
0 & 1 & 0 & \ldots & 0 & -c_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & -c_{n-2} \\
0 & 0 & 0 & \ldots & 1 & -c_{n-1}
\end{bmatrix}.
\]

3. Let $V$ be a finite dimensional vector space over the field $F$. Let $V^*$ be the dual space of $V$ (that is, $V^*$ is the vector space of linear transformations $T : V \to F$). Show that $V \cong V^*$. 


II. Group Theory

1. Let $G$ be a group, and let $g \in G$ be an element of order greater than 2 (possibly infinite) such that the conjugacy class of $g$ in $G$ has an odd number of elements. Prove that $g$ is not conjugate to $g^{-1}$.

2. Let $N$ be a normal subgroup of the group $G$. Show that if $N \cap G'' = 1$, then $N$ is contained in the center of $G$.

3. Let $G$ be a group with a normal subgroup $N$ of order 5, such that $G/N \cong S_3$. Show that $|G| = 30$, $G$ has a normal subgroup of order 15, and $G$ has 3 subgroups of order 10 that are not normal.

4. Let $n$ be a positive integer and let $A = \mathbb{Z}^n$. Prove that if $B$ is any subgroup of $A$ that is generated by fewer than $n$ elements, then the index $|A : B|$ is infinite.

5. Show that a simple group of order 168 must be isomorphic to a subgroup of the alternating group $A_8$. 
III. Ring Theory

1. Let $R$ be the ring of all $2 \times 2$ matrices of the form \[
\begin{pmatrix}
a & b \\
-b & a
\end{pmatrix}
\] where $a$ and $b$ are real numbers. Prove that $R$ is isomorphic to $\mathbb{C}$, the field of complex numbers.

2. Let $R$ be a commutative ring with identity. Suppose $R$ contains an idempotent element $a$ other than 0 or 1. Show that every prime ideal in $R$ contains an idempotent element other than 0 or 1. (An element $a \in R$ is an idempotent if $a^2 = a$.)

3. Show that if $p$ is a prime such that $p \equiv 1 \pmod{4}$, then $\mathbb{Z}[\sqrt{-p}]$ is not a unique factorization domain.

4. Let $R$ be a ring with identity such that the identity map is the only ring automorphism of $R$. Prove that the set $N$ of all nilpotent elements of $R$ is an ideal of $R$.

5. Let $R$ be a commutative ring with identity. Prove that any non-empty set of prime ideals of $R$ contains maximal and minimal elements.

IV. Field Theory

1. Show that $p(x) = x^3 + x - 6$ is irreducible over $\mathbb{Q}[\sqrt{-1}]$.

2. Let $f(x)$ and $g(x)$ be irreducible polynomials in $F[x]$ of degrees $m$ and $n$, respectively, where $(m, n) = 1$. Show that if $a$ is a root of $f(x)$ in some field extension of $F$, then $g(x)$ is irreducible in $F(a)[x]$.

3. Let $K$ be the splitting field of $x^2 + 2$ over $\mathbb{Q}$. Prove or disprove that $i = \sqrt{-1}$ is an element of $K$.

4. Show that every finite field is perfect. A field $F$ is called perfect if every element of an algebraic closure of $F$ is separable over $F$.

5. Determine the Galois group of $x^3 + 3x^2 - 1$ over $\mathbb{Q}$. 

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