QUALIFYING EXAM IN ALGEBRA

February 2021

1. There are 18 problems on the exam. Work and turn in 10 problems, in the following categories.

I.	Linear Algebra	—	1 problem
II.	Group Theory	—	3 problems
III.	Ring Theory	—	2 problems
IV.	Field Theory	—	3 problems
Any of the four areas		—	1 problem

2. Turn in only 10 problems. No credit will be given for extra problems. All problems are weighted equally.

3. Put each problem on a separate sheet of paper, and write only on one side. Put your name on each page.

4. If you feel there is a misprint or error in the statement of a problem, then interpret it in such a way that the problem is not trivial.

I. Linear Algebra

1. Let A be a complex n by n matrix with characteristic polynomial $f(x)$ and minimal polynomial $g(x)$. By the Cayley-Hamilton Theorem we know that $f(A) = 0$. Prove that $g(x)$ divides $f(x)$ and that $f(x)$ divides some power of $g(x)$.

2. Let A and B be complex 3 by 3 matrices having the same eigenvectors. Suppose the minimal polynomial of A is $(x-1)^2$ and the characteristic polynomial of B is x^3. Show that the minimal polynomial of B is x^2.

3. A linear transformation $T : V \to W$ is said to be independence preserving if $T(I) \subseteq W$ is linearly independent whenever $I \subseteq V$ is a linearly independent set. Show that T is independence preserving if and only if T is one-to-one.

II. Group Theory

1. Let A and B be subgroups of a finite group G. Prove the inequality of indices:

$$|A : A \cap B| \leq |G : B|$$

and show that this inequality turns into equality if and only if $G = AB$.

2. Let N be a nontrivial normal subgroup of the p-group P. Prove that $N \cap Z(P)$ is nontrivial.

3. Prove that a group of order $29 \cdot 30$ has a normal Sylow 29-subgroup.

4. Show that a group order $2021 = 43 \cdot 47$ is solvable.

5. Let M be a maximal subgroup of the finite group G. Show that if M is normal in G, then the index $|G : M|$ is a prime number.

III. Ring Theory

1. Let R be a commutative ring in which any two ideals I and J are comparable (that is, $I \subseteq J$ or $J \subseteq I$). Prove that every finitely generated ideal is principal.

2. Let R be a commutative ring with 1. Show that an ideal P of R is prime if and only if R/P is an integral domain.

3. In a commutative ring R with 1, prove that if u is a unit and n is nilpotent, then $u+n$ is a unit.

4. Explain why $\mathbb{Z}_3[x]/\langle x^3+x^2+2\rangle$ is a field and find the multiplicative inverse of x^2+1 in $\mathbb{Z}_3[x]/\langle x^3+x^2+2\rangle$.

5. Let R be a ring satisfying the descending chain condition on right ideals. If R is not a division ring, prove that R contains nontrivial zero divisors.

IV. Field Theory

1. Find the minimal polynomial of $\sqrt[3]{2+\sqrt{2}}$ over the field $\mathbb{Q}$ of rational numbers, and prove it is the minimal polynomial.

2. Let $F \subset E \subset K$ be a tower of fields such that $K = F(\alpha)$ is an algebraic extension of F and $F \neq E$. Prove that the minimal polynomial of α over F is reducible in $E[x]$.

3. (a) Determine the Galois group of x^4-4 over the field $\mathbb{Q}$ of rational numbers.

   (b) How many intermediate fields are there between $\mathbb{Q}$ and the splitting field of x^4-4?

4. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of degree 4 with exactly two real roots. Show that the Galois group of f has order either 24 or 8.

5. Let $K = \mathbb{Z}_3(i)$ and let $f(x) = x^4+x^3+x+2 \in \mathbb{Z}_3[x]$.

   (a) Show that f splits over K.

   (b) Find a generator α of the multiplicative group of K and express the roots of f in terms of α.