QUALIFYING EXAM IN ALGEBRA

January 2022

1. There are 18 problems on the exam. Work and turn in 10 problems, in the following categories.

I.	Linear Algebra	—	1 problem
II.	Group Theory	—	3 problems
III.	Ring Theory	—	2 problems
IV.	Field Theory	—	3 problems
Any of the four areas		—	1 problem

2. Turn in only 10 problems. No credit will be given for extra problems. All problems are weighted equally.

3. Put each problem on a separate sheet of paper, and write only on one side. Put your name on each page.

4. If you feel there is a misprint or error in the statement of a problem, then interpret it in such a way that the problem is not trivial.

I. Linear Algebra

1. Let A be a matrix of the form

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ c_1 & c_2 & c_3 & \cdots & c_n \end{bmatrix}.$$

   Show that the minimal polynomial and characteristic polynomial of A are equal.

2. Let F be a field and V a finite dimensional vector space over F with $\dim V > 1$. Suppose $f : V \to V$ and $g : V \to V$ are distinct nilpotent linear transformations satisfying $f^2 = g^2 = 0$ and that the only subspaces of V that are both f-invariant and g-invariant are V and $\{0\}$. Prove the following:
   (a) The image of f equals the null space of f and the image of g equals the null space of g.
   (b) V is the direct sum of the null spaces of f and g.
   (c) $\dim V$ is even.

3. Let $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ be a basis for a vector space V over $\mathbb{R}$. Show that if $\boldsymbol{w}$ is any vector in V, then for some choice of sign $\pm$, $\{\boldsymbol{v}_1 \pm \boldsymbol{w}, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ is a basis for V.

II. Group Theory

1. Show that if $\mathcal{K}$ and $\mathcal{L}$ are conjugacy classes of groups G and H, respectively, then $\mathcal{K} \times \mathcal{L}$ is a conjugacy class of $G \times H$.

2. Let G be a finite simple group with a subgroup H of prime index p. Show that p must be the largest prime dividing the order of G.

3. Let H be a proper subgroup of the finite group G. Prove that the union of all the conjugates of H is a proper subset of G.

4. Determine, up to isomorphism, the groups of order $2022 = 2 \cdot 3 \cdot 337$.

5. Let H be a normal subgroup of G, $K \leqslant H$, and assume every automorphism of H is inner. Prove that $G = HN_G(K)$, where $N_G(K)$ is the normalizer of K in G.

III. Ring Theory

1. Denote the set of invertible elements of the ring $\mathbb{Z}_n$ by U_n.
   (a) List all the elements of U_{20}.
   (b) Is U_{20} a cyclic group under multiplication? Justify your answer.

2. Let R be any ring with identity, and n any positive integer. Prove that if $M_n(R)$ is the ring of $n \times n$ matrices with entries in R, then $M_n(I)$ is an ideal of $M_n(R)$ whenever I is an ideal of R and every ideal of $M_n(R)$ has this form.

3. Let R be an integral domain. Show that if all prime ideals of R are principal, then R is a Principal Ideal Domain.

4. Let R be a commutative ring with identity that has exactly one prime ideal P. Prove the following:
   (a) R/P is a field.
   (b) R is isomorphic to R_P, the ring of quotients of R with respect to the multiplicative set $R - P = \{s \in R \mid s \notin P\}$.

5. Let D be an integral domain and $D[x]$ the polynomial ring over D.
   Suppose $\varphi : D[x] \to D[x]$ is an isomorphism such that $\varphi(d) = d$ for all $d \in D$.
   Show that $\varphi(x) = ax + b$ for some $a, b \in D$ and that a is a unit of D.

IV. Field Theory

1. Let p and q be distinct primes. Prove that $\sqrt{q}$ does not belong to $\mathbb{Q}(\sqrt{p})$.

2. Let F be a field and $f(x) \in F[x]$ an irreducible polynomial. Prove that there is a prime p, an integer $a \geqslant 0$ and a separable polynomial $g(x) \in F[x]$ such that $f(x) = g(x^{p^a})$.

3. Let F be any field and let $f(x) = x^n - 1 \in F[x]$. Show that if K is the splitting field of $f(x)$ over F, then K is separable over F (hence Galois) and $\mathrm{Gal}(K/F)$ is abelian.

4. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of degree 5. Assume $f(x)$ has exactly 3 distinct real roots and one complex conjugate pair of roots. Prove that if K is the splitting field of $f(x)$ over $\mathbb{Q}$, then $\mathrm{Gal}(K/\mathbb{Q})$ is S_5.

5. Let F be a finite field. Show that the product of all the non-zero elements of F is -1.