QUALIFYING EXAM IN ALGEBRA

January 2023

1. There are 18 problems on the exam. Work and turn in 10 problems, in the following categories.

I.	Linear Algebra	—	1 problem
II.	Group Theory	—	3 problems
III.	Ring Theory	—	2 problems
IV.	Field Theory	—	3 problems
Any of the four areas		—	1 problem

2. Turn in only 10 problems. No credit will be given for extra problems. All problems are weighted equally.

3. Put each problem on a separate sheet of paper, and write only on one side. Put your name on each page.

4. If you feel there is a misprint or error in the statement of a problem, then interpret it in such a way that the problem is not trivial.

I. Linear Algebra

1. Does there exist a 2023×2023 real matrix A such that $A^2 = -I$, where I is the identity matrix?

2. (a) Show that two 3×3 complex matrices are similar if and only if they have the same characteristic and minimal polynomials.

   (b) Is the conclusion of part (a) true for larger matrices? Prove or give a counter-example.

3. Let V be a finite dimensional vector space over a field F and let $T : V \to V$ be a nilpotent linear transformation. Show that the trace of T is 0.

II. Group Theory

1. Let G be a group, $g \in G$ an element of order greater than 2 (possibly infinite) such that the conjugacy class of g has an odd number of elements. Prove that g is not conjugate to g^{-1}.

2. Let G be a group with a normal subgroup N of order 5, such that $G/N \cong S_3$. Show that $|G| = 30$, G has a normal subgroup of order 15, and G has 3 subgroups of order 10 that are not normal.

3. Let G be a group with a proper subgroup of finite index. Show that G has a proper normal subgroup of finite index.

4. Let G act on a set Ω and assume N is a normal subgroup of G that is contained in the kernel of the action. Show that there is a natural action of G/N on Ω which satisfies the property that G is transitive if and only if G/N is transitive.

5. Show that a group of order $3 \cdot 5 \cdot 7$ has a normal Sylow 7-subgroup and a central Sylow 5-subgroup.

III. Ring Theory

1. Give an example of a commutative ring R and ideals I and J in which $I \cdot J \neq I \cap J$. Also, prove that if $I + J = R$, then necessarily $I \cdot J = I \cap J$.

2. Let $f : \mathbb{Z}_n \to \mathbb{Z}_n$ be a map such that $f(a+b) = f(a) + f(b)$ and $f(ab) = f(a)b + af(b)$ for all $a, b \in \mathbb{Z}_n$. Prove that f is identically zero.

3. Let $D = \mathbb{Z}(\sqrt{13}) = \{m + n\sqrt{13} \mid m, n \in \mathbb{Z}\}$ and $F = \mathbb{Q}(\sqrt{13})$, the field of fractions of D. Show the following:
   (a) $x^2 + 3x - 1$ is irreducible in $D[x]$ but not in $F[x]$.
   (b) D is not a unique factorization domain.

4. Let a be an element of a ring R. Suppose that there exists a polynomial $p(x) \in R[x]$ such that $ax \cdot p(x) = ax + p(x)$. Prove that a is nilpotent.

5. Define Noetherian ring and prove that if R is Noetherian, then $R[x]$ is Noetherian.

IV. Field Theory

1. Let K be a finite degree extension of the field F such that $[K : F]$ is relatively prime to 6. Show that if $u \in K$, then $F(u) = F(u^3)$.

2. Let K be the splitting field of $x^2 + 2$ over $\mathbb{Q}$. Prove or disprove that $i = \sqrt{-1}$ is an element of K.

3. Let F be a field of characteristic p and let x be an indeterminate over F.
   (a) Show that $F(x^p)$ is a proper subfield of $F(x)$.
   (b) Show that $F(x)$ is a splitting field for some polynomial over $F(x^p)$.

4. Find, with proof, the Galois group of the splitting field over the rational numbers of the polynomial $f(x) = x^6 + 3$.

5. Let F be a finite field. Prove that the polynomial ring $F[x]$ contains irreducible polynomials of arbitrarily large degree.