QUALIFYING EXAM IN ALGEBRA

JANUARY 1992

1. Work as many problems as you can. It is to your advantage to demonstrate a broad background.

2. If you feel there is a misprint or error in the statement of the problem, then interpret it in such a way that the problem is not trivial.

Group Theory

1. (a) Find the centralizer in S_7 of $(\,1\;\;2\;\;3\,)(\,4\;\;5\;\;6\;\;7\,)$.

   (b) How many elements of order 12 are there in S_7?

2. Let $f : G \to H$ be a homomorphism of groups with kernel K and image I.

   (a) Show that if N is a subgroup of G then $f^{-1}(f(N)) = KN$.

   (b) Show that if L is a subgroup of H then $f(f^{-1}(L)) = I \cap L$.

3. Let G be a finite group.

   (a) Show that every proper subgroup of G is contained in a maximal subgroup.

   (b) Show that the intersection of all maximal subgroups of G is a normal subgroup.

4. Let N be a group with trivial center such that all automorphisms of N are inner automorphisms. Show that whenever N occurs as a normal subgroup of a group G, there is a subgroup H of G such that $G = H \times N$.

5. Let G be a subgroup of the symmetric group S_n. Show that if G contains an odd permutation then $G \cap A_n$ is of index 2 in G.

6. Show that a simple group of order 168 must be isomorphic to a subgroup of the alternating group A_8.

Ring Theory

1. Let p be a prime and let $F_p = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \mid a, b \in \boldsymbol{Z}_p \right\}$.

   (a) Show that F_p, with the usual matrix operations, is a commutative ring with identity.

   (b) Show that F_7 is a field.

   (c) Show that F_{13} is not a field.

2. Let p be a prime.

   (a) Show that if $p \equiv 1 (\text{mod } 4)$, then $x^2 + 1$ is not irreducible in $\boldsymbol{Z}_p[x]$.

   (b) Show that if $p \equiv 3 (\text{mod } 4)$, then $x^2 + 1$ is irreducible in $\boldsymbol{Z}_p[x]$.

3. Let R be a commutative ring with 1 such that for every x in R there is an integer $n > 1$ (depending on x) such that $x^n = x$. Show that every prime ideal of R is maximal.

4. Let $D = \boldsymbol{Z}(\sqrt{13}) = \{\, m + n\sqrt{13} \,|\, m, n \in \boldsymbol{Z} \,\}$ and $F = \boldsymbol{Q}(\sqrt{13})$ its field of fractions. Show the following:

   (a) $x^2 + 3x - 1$ is irreducible in $D[x]$ but not in $F[x]$.

   (b) D is not a unique factorization domain.

5. Let R be a ring.

   (a) Show that there is a unique smallest (with respect to inclusion) ideal A such that R/A is a commutative ring.

   (b) Give an example of a ring R such that for every proper ideal I, R/I is not commutative. Verify your example.

   (c) For the ring $R = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c \in \boldsymbol{Z} \right\}$ with the usual matrix operations, find the ideal A of part (a).

6. Let R be a non-zero commutative ring with 1.

   (a) Let S be a multiplicative subset of R not containing 0 and let P be maximal in the set of ideals of R not intersecting S. Show that P is a prime ideal.

   (b) Show that the set of nilpotent elements of R is the intersection of all prime ideals.

Field Theory

1. Let K be an extension field of the field F such that $[K : F]$ is odd. Show that if $u \in K$ then $F(u) = F(u^2)$.

2. Let $F \subset E \subset K$ be a tower of fields such that $K = F(\alpha)$ with α algebraic over F. Prove that if $f(x) \in F[x]$ is the minimal polynomial of α over F and $F \neq E$, then $f(x)$ is not irreducible in $E[x]$.

3. Let $f(x) \in \boldsymbol{Q}[x]$ be an irreducible polynomial of degree n with roots $\alpha_1, \ldots, \alpha_n$. Show that $\sum_{i=1}^{n} \frac{1}{\alpha_i}$ is a rational number.

4. Let $f(x) = x^4 + x^3 + 4x - 1 \in \boldsymbol{Z}_5[x]$.
   Find the Galois group of the splitting field of f over $\boldsymbol{Z}_5$.

5. Let η be a complex primitive 11-th root of unity and let $K = \boldsymbol{Q}(\eta)$. Find $\mathrm{Gal}(K/\boldsymbol{Q})$ and express each intermediate field F between K and $\boldsymbol{Q}$ as $F = \boldsymbol{Q}(\beta)$ for some $\beta \in K$.

6. Let $f(x) \in \boldsymbol{Q}[x]$ be an irreducible polynomial of degree 4 with exactly 2 real roots. Show that the Galois group of f is either S_4 or the dihedral group of order 8.

Linear Algebra

1. Let V and W be finite dimensional vector spaces and let $T : V \to W$ be a linear transformation. Show that $\dim(\ker T) + \dim(\mathrm{Im}\ T) = \dim(V)$.

2. Let V be a finite dimensional vector space over the field F. Let V^* be the dual space of V (that is, V^* is the vector space of linear transformations $T : V \to F$). Show that $V \cong V^*$.