QUALIFYING EXAM IN ALGEBRA

January 1993

1. Work as many problems as you can. It is to your advantage to demonstrate a broad background.

2. If you feel there is a misprint or error in the statement of the problem, then interpret it in such a way that the problem is not trivial.

Group Theory

1. (a) Find the centralizer in S_7 of $(\,1\ \ 2\,)(\,3\ \ 4\ \ 5\ \ 6\ \ 7\,)$.

   (b) How many elements of order 10 are there in S_7? How many are in A_7?

2. Let P be a p-group and let H be a proper subgroup of P. Prove that H is a proper subgroup of its normalizer $N_P(H)$.

3. Let p, q be distinct primes. Show that a group of order p^2q has a normal Sylow p-subgroup or a normal Sylow q-subgroup.

4. Let H be a subgroup of index n in a group G. Let S_n be the symmetric group on n letters and let $S_{n-1} \subseteq S_n$ be the usual embedding. Show that $H = f^{-1}(S_{n-1})$ for some homomorphism $f : G \to S_n$. (Hint: Let G act on the cosets of H.)

5. Let $GL_2(\boldsymbol{C})$ be the group (under multiplication) of invertible 2×2 matrices with complex entries. Give a complete list of conjugacy class representatives for $GL_2(\boldsymbol{C})$.

Ring Theory

1. Let R_1 and R_2 be commutative rings with identities and let $R = R_1 \times R_2$. Show that every ideal I of R is of the form $I = I_1 \times I_2$ with I_i an ideal of R_i for $i = 1, 2$.

2. Let m and n be relatively prime integers.

   (a) Show that if c and d are any integers, then there is an integer x such that $x \equiv c (\mathrm{mod}\ m)$ and $x \equiv d (\mathrm{mod}\ n)$.

   (b) Show that $\boldsymbol{Z}_{mn}$ and $\boldsymbol{Z}_m \times \boldsymbol{Z}_n$ are isomorphic as rings.

3. A *local* ring is a commutative ring with 1 which has a unique maximal ideal.
   Show that a ring R is local if and only if the set of non-units in R is an ideal.

4. Show that if R is a commutative Noetherian ring with identity, then the polynomial ring $R[x]$ is also Noetherian.

Field Theory

1. Let η be a complex primitive 7th root of unity and let $K = \boldsymbol{Q}(\eta)$, where $\boldsymbol{Q}$ is the field of rational numbers. Show that there is a unique extension F of degree 2 of $\boldsymbol{Q}$ contained in K and find $q \in \boldsymbol{Q}$ such that $F = \boldsymbol{Q}(\sqrt{q})$.

2. Show that if a field K is an algebraic extension of a field F and $\sigma : K \to K$ is an F-monomorphism, then σ is onto.

3. Let E be a finite dimensional Galois extension of a field F and let $G = Gal(E/F)$. Suppose that G is an abelian group. Prove that if K is any field between E and F, then K is a Galois extension of F.

4. Let K be a finite dimensional extension field of L and let $\sigma : L \to F$ be an embedding of L into a field F. Prove that there are at most $|K : L|$ extensions of σ to embeddings of K into F.

Linear Algebra

1. Let A be a square complex matrix with a single eigenvalue λ. Show that the number of blocks in the Jordan form of A is the dimension of the λ-eigenspace.

2. Let V and W be finite dimensional vector spaces and let $T : V \to W$ be a linear transformation. Show that $\dim(\ker T) + \dim(\text{Im } T) = \dim(V)$.