

## QUALIFYING EXAM IN ALGEBRA

January 1994

1. Work as many problems as you can. It is to your advantage to demonstrate a broad background.
2. If you feel there is a misprint or error in the statement of the problem, then interpret it in such a way that the problem is not trivial.

## Linear Algebra

1. Let  $V$  be a finite dimensional vector space over the field  $F$ . Let  $V^*$  be the dual space of  $V$  (that is,  $V^*$  is the vector space of linear transformations  $T : V \rightarrow F$ ). Show that  $V \cong V^*$ .
2. Find all possible Jordan canonical forms of a  $5 \times 5$  complex matrix with minimal polynomial  $(x - 2)^2(x - 1)$ .

## Group Theory

1. (a) Find the centralizer in  $S_8$  of  $(1\ 2\ 3)(4\ 5\ 6\ 7\ 8)$ .  
(b) How many elements of order 15 are there in  $S_8$ ?
2. A group  $N$  is said to be *complete* if the center of  $N$  is trivial and every automorphism of  $N$  is inner. Show that if  $G$  is a group,  $N \trianglelefteq G$ , and  $N$  is complete, then  $G = N \times C_G(N)$ .
3. Let  $H$  and  $K$  be subgroups of a group  $G$  with  $K \trianglelefteq G$ . Show that if  $H$  and  $K$  are solvable, then  $HK$  is solvable.
4. Show that a group of order 96 must have a normal subgroup of order 16 or 32.
5. Show that if  $A$ ,  $B$ , and  $C$  are abelian groups, then

$$\text{Hom}(A, B \oplus C) \cong \text{Hom}(A, B) \oplus \text{Hom}(A, C).$$

## Ring Theory

1. Find all values of  $a$  in  $\mathbf{Z}_3$  such that the quotient ring

$$\mathbf{Z}_3[x]/(x^3 + x^2 + ax + 1)$$

is a field. Justify your answer.

2. Let  $R_1$  and  $R_2$  be commutative rings with identities. The cartesian product  $R_1 \times R_2$  is a ring under componentwise addition and multiplication (you need not prove this). Show that every ideal  $I$  of  $R_1 \times R_2$  is of the form  $I_1 \times I_2$ , where  $I_1$  is an ideal of  $R_1$  and  $I_2$  is an ideal of  $R_2$ .
3. Let  $R$  be a commutative ring with identity and let  $S$  be the set of all elements of  $R$  that are *not* zero-divisors. Show that there is a prime ideal  $P$  such that  $P \cap S$  is empty. (Hint: Use Zorn's Lemma.)
4. Let  $R$  be a ring and  $M$  a simple  $R$ -module. Let  $D = \text{End}_R(M)$  be the ring of  $R$ -endomorphisms of  $M$  (under composition and pointwise addition). Prove that  $D$  is a division ring.

## Field Theory

1. Let  $E$  be an extension field of a field  $F$  and  $f(x), g(x) \in F[x]$ . Prove that a greatest common divisor of  $f$  and  $g$  in  $F[x]$  is also a greatest common divisor of  $f$  and  $g$  in  $E[x]$ .
2. Let  $\alpha_1, \alpha_2$ , and  $\alpha_3$  be real numbers such that  $(\alpha_i)^2 \in \mathbf{Q}$  for each  $i$ , and let  $K = \mathbf{Q}(\alpha_1, \alpha_2, \alpha_3)$ . Show that  $\sqrt[3]{2}$  is not in  $K$ .
3. Let  $F \subseteq L \subseteq K$  with  $[L : F]$  finite, and let  $\alpha$  be an element of  $K$ . Show that  $\alpha$  is algebraic over  $L$  if and only if  $\alpha$  is algebraic over  $F$ .
4. (a) Show that for every prime  $p$  and every positive integer  $n$  there is an irreducible polynomial of degree  $n$  over the field  $\mathbf{F}_p$  of  $p$  elements.  
(b) Show that for every positive integer  $n$  there is an irreducible polynomial of degree  $n$  over the field  $\mathbf{Q}$  of rational numbers.
5. (a) Find the Galois group of  $x^3 - 2$  over  $\mathbf{Q}$  and demonstrate the Galois correspondence between the subgroups of the Galois group and the subfields of the splitting field.  
(b) Find all automorphisms of  $\mathbf{Q}(\sqrt[3]{2})$ . Is there an  $f(x) \in \mathbf{Q}[x]$  with splitting field  $\mathbf{Q}(\sqrt[3]{2})$ ? Explain.