QUALIFYING EXAM IN ALGEBRA

January 1997

1. There are 17 problems on the exam. Work and turn in 10 problems, in the following categories.

I. Linear Algebra	—	1 problem
II. Group Theory	—	3 problems
III. Ring Theory	—	2 problems
IV. Field Theory	—	3 problems
Any of the four areas	—	1 problem

2. Turn in only 10 problems. No credit will be given for extra problems. All problems are weighted equally.

3. Put each problem on a separate sheet of paper, and write only on one side. Put your name on each page.

4. If you feel there is a misprint or error in the statement of a problem, then interpret it in such a way that the problem is not trivial.

I. Linear Algebra

1. Let V be a finite dimensional vector space over a field F and let $T : V \to V$ be a nilpotent linear transformation. Show that the trace of T is 0.

2. Let A be a square complex matrix with a single eigenvalue λ. Show that the number of blocks in the Jordan form of A is the dimension of the λ-eigenspace.

3. Let V be the vector space over the field $\boldsymbol{R}$ of real numbers consisting of all functions from $\boldsymbol{R}$ into $\boldsymbol{R}$. Let U be the subspace of even functions and W the subspace of odd functions. Show that $V = U \oplus W$.

II. Group Theory

1. (a) Write $\sigma = (\ 4\ \ 5\ \ 6\ )(\ 2\ \ 3\ )(\ 1\ \ 2\ )(\ 6\ \ 7\ \ 8\ )$ as a product of disjoint cycles and find the order of σ.

   (b) Let $n > 1$ be an odd integer. Show that S_n has an element of order $2(n-2)$.

2. Let H be a subgroup of the group G with the property that whenever two elements of G are conjugate, then the conjugating element can be chosen within H. Prove that the commutator subgroup G' of G is contained in H.

3. Let G be a finite non-solvable group, each of whose proper subgroups is solvable. Let $\Phi(G)$ be the Frattini subgroup of G (i.e., the intersection of all maximal subgroups of G). Show that $G/\Phi(G)$ is a non-abelian simple group.

4. (Frattini argument) Let G be a finite group and H a normal subgroup. Show that if P is a Sylow p-subgroup of H, then $G = HN_G(P)$.

5. Show that a group of order $180 = 2^2 \cdot 3^2 \cdot 5$ cannot be simple. [Hint: First show that S_6 has no simple subgroup of index 4 (i.e. order 180).]

III. Ring Theory

1. Give an example of each of the following.
   (a) An infinite non-commutative ring with only finitely many ideals.
   (b) A unique factorization domain that is not a principal ideal domain.
   (c) A finite non-commutative ring.
   (d) A non-zero prime ideal of a commutative ring that is not a maximal ideal.

2. Let D be a principal ideal domain.
   (a) For $a, b \in D$, define a *least common multiple* of a and b.
   (b) Show that $d \in D$ is a least common multiple of a and b if and only if $(a) \cap (b) = (d)$.

3. Let D be a principal ideal domain and let P be a non-zero prime ideal. Show that D_P, the localization of D at P, is a principal ideal domain and has a unique irreducible element, up to associates.

4. Let R be a commutative ring with identity. An ideal I of R is *irreducible* if it cannot be expressed as the intersection of two proper ideals of R. Show the following.
   (a) If P is a prime ideal, then P is irreducible.
   (b) If x is a non-zero element of R, then there is an ideal I_x, maximal with respect to the property that $x \notin I_x$.
   (c) The ideal I_x from part (b) is irreducible.

IV. Field Theory

1. Let K be an extension field of F and let α be an element of K. Show that if $F(\alpha) = F(\alpha^2)$, then α is algebraic over F.

2. Let F be a field and let $f(x) = x^n - x \in F[x]$. Show that if char $F = 0$ or if char $F = p$ and $p \not| n - 1$, then f has no multiple root in any extension of F.

3. Let $\boldsymbol{F}_p$ be the field of p elements and let K be an extension of $\boldsymbol{F}_p$ of degree n. Show that the set of subfields of K is linearly ordered (i.e., for every pair of subfields L_1, L_2, either $L_1 \subseteq L_2$ or $L_2 \subseteq L_1$) if and only if n is a prime power.

4. Let K be a Galois extension of F with $|\mathrm{Gal}(K/F)| = 12$. Prove that there exists a subfield E of K containing F with $[E : F] = 3$. Does a subextension necessarily exist satisfying $[E : F] = 2$? Explain.

5. Let K be a splitting field for $x^5 - 2$ over the field $\boldsymbol{Q}$ of rational numbers.
   (a) Determine $[K : \boldsymbol{Q}]$.
   (b) Show that $\mathrm{Gal}(K/\boldsymbol{Q})$ is non-abelian.