Boyce/DiPrima 9th ed, Ch 2.1: Linear Equations; Method of Integrating Factors

- A linear first order ODE has the general form

$$\frac{dy}{dt} = f(t, y)$$

where *f* is linear in *y*. Examples include equations with constant coefficients, such as those in Chapter 1,

$$y' = -ay + b$$

or equations with variable coefficients:

$$\frac{dy}{dt} + p(t)y = g(t)$$

Constant Coefficient Case

- For a first order linear equation with constant coefficients,

$$y' = -ay + b,$$

recall that we can use methods of calculus to solve:

$$\frac{dy/dt}{y - b/a} = -a$$

$$\int \frac{dy}{y - b/a} = -\int a\,dt$$

$$\ln\left|y - b/a\right| = -a\,t + C$$

$$y = b/a + ke^{at},\;\; k = \pm e^{C}$$

Variable Coefficient Case: Method of Integrating Factors

- We next consider linear first order ODEs with variable coefficients:

$$\frac{dy}{dt} + p(t)y = g(t)$$

- The method of integrating factors involves multiplying this equation by a function $\mu(t)$, chosen so that the resulting equation is easily integrated.

Example 1: Integrating Factor (1 of 2)

- Consider the following equation:

$$y' + \tfrac{1}{2}y = \tfrac{1}{2}e^{t/3}$$

- Multiplying both sides by $\mu(t)$, we obtain

$$\mu(t)\frac{dy}{dt} + \frac{1}{2}\mu(t)y = \frac{1}{2}e^{t/3}\mu(t)$$

- We will choose $\mu(t)$ so that left side is derivative of known quantity. Consider the following, and recall product rule:

$$\frac{d}{dt}\left[\mu(t)y\right] = \mu(t)\frac{dy}{dt} + \frac{d\mu(t)}{dt}y$$

- Choose $\mu(t)$ so that

$$\mu'(t) = \frac{1}{2}\mu(t) \;\;\Rightarrow\;\; \mu(t) = e^{t/2}$$

Example 1: General Solution (2 of 2)

- With $\mu(t) = e^{t/2}$, we solve the original equation as follows:

$$y' + \frac{1}{2}y = \frac{1}{2}e^{t/3}$$

$$e^{t/2}\frac{dy}{dt} + \frac{1}{2}e^{t/2}y = \frac{1}{2}e^{5t/6}$$

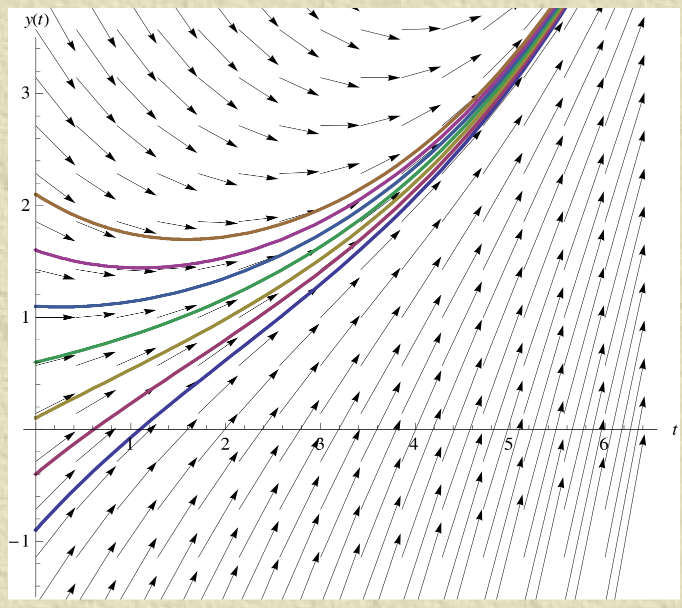
$$\frac{d}{dt}\left[e^{t/2}y\right] = \frac{1}{2}e^{5t/6}$$

$$e^{t/2}y = \frac{3}{5}e^{5t/6} + C$$

general solution:

$$y = \frac{3}{5}e^{t/3} + Ce^{-t/2}$$

Sample Solutions : $y = \frac{3}{5}e^{t/3} + Ce^{-t/2}$



Method of Integrating Factors: Variable Right Side

- In general, for variable right side $g(t)$, the solution can be found as follows:

$$y' + ay = g(t)$$

$$\mu(t)\frac{dy}{dt} + a\mu(t)\,y = \mu(t)\,g(t)$$

$$e^{at}\frac{dy}{dt} + ae^{at}\,y = e^{at}\,g(t)$$

$$\frac{d}{dt}\left[e^{at}\,y\right] = e^{at}\,g(t)$$

$$e^{at}\,y = \int e^{at}\,g(t)\,dt$$

$$y = e^{-at}\int e^{at}\,g(t)\,dt + Ce^{-at}$$

Example 2: General Solution (1 of 2)

- We can solve the following equation

$$y' - 2y = 4 - t$$

using the formula derived on the previous slide:

$$y = e^{-at}\int e^{at} g(t)dt + Ce^{-at} = e^{2t}\int e^{-2t}(4-t)dt + Ce^{2t}$$

- Integrating by parts, $\int e^{-2t}(4-t)dt = \int 4e^{-2t}dt - \int te^{-2t}dt$

$$= -2e^{t/5} - \left[-\frac{1}{2}te^{-2t} + \int \frac{1}{2}e^{-2t}dt\right]$$

$$= -\frac{7}{4}e^{-2t} + \frac{1}{2}te^{-2t}$$

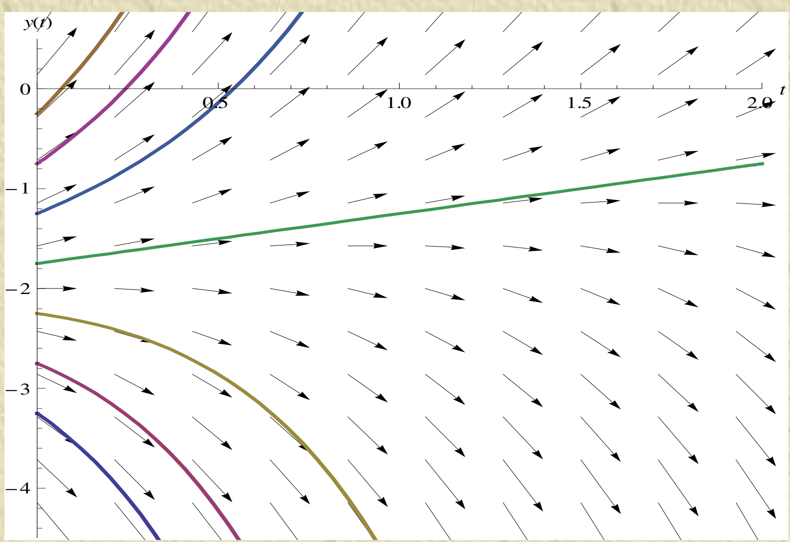
- Thus $y = e^{2t}\left(-\frac{7}{4}e^{-2t} + \frac{1}{2}te^{-2t}\right) + Ce^{2t} = -\frac{7}{4} + \frac{1}{2}t + Ce^{2t}$

$$y' - 2y = 4 - t$$

Example 2: Graphs of Solutions (2 of 2)

- The graph shows the direction field along with several integral curves. If we set $C = 0$, the exponential term drops out and you should notice how the solution in that case, through the point (0, -7/4), separates the solutions into those that grow exponentially in the positive direction from those that grow exponentially in the negative direction..

$$y = -\frac{7}{4} + \frac{1}{2}t + Ce^{2t}$$



Method of Integrating Factors for General First Order Linear Equation

- Next, we consider the general first order linear equation

$$y' + p(t)y = g(t)$$

- Multiplying both sides by $\mu(t)$, we obtain

$$\mu(t)\frac{dy}{dt} + p(t)\mu(t)y = g(t)\mu(t)$$

- Next, we want $\mu(t)$ such that $\mu'(t) = p(t)\mu(t)$, from which it will follow that

$$\frac{d}{dt}\left[\mu(t)y\right] = \mu(t)\frac{dy}{dt} + p(t)\mu(t)y$$

Integrating Factor for General First Order Linear Equation

- Thus we want to choose $\mu(t)$ such that $\mu'(t) = p(t)\mu(t)$.
- Assuming $\mu(t) > 0$, it follows that

$$\int \frac{d\mu(t)}{\mu(t)} = \int p(t)dt \quad \Rightarrow \quad \ln \mu(t) = \int p(t)dt + k$$

- Choosing $k = 0$, we then have

$$\mu(t) = e^{\int p(t)dt},$$

and note $\mu(t) > 0$ as desired.

Solution for General First Order Linear Equation

- Thus we have the following:

$$y' + p(t)y = g(t)$$

$$\mu(t)\frac{dy}{dt} + p(t)\mu(t)y = \mu(t)g(t), \quad \text{where } \mu(t) = e^{\int p(t)dt}$$

- Then

$$\frac{d}{dt}[\mu(t)y] = \mu(t)g(t)$$

$$\mu(t)y = \int \mu(t)g(t)dt + c$$

$$y = \frac{\int \mu(t)g(t)dt + c}{\mu(t)}, \quad \text{where } \mu(t) = e^{\int p(t)dt}$$

Example 3: General Solution (1 of 2)

- To solve the initial value problem

$$ty' + 2y = 4t^2, \quad y(1) = 2,$$

first put into standard form:

$$y' + \frac{2}{t}y = 4t, \quad \text{for } t \neq 0$$

- Then

$$\mu(t) = e^{\int p(t)dt} = e^{\int \frac{2}{t}dt} = e^{2\ln|t|} = e^{\ln\left(t^2\right)} = t^2$$

and hence

$$y = \frac{\int \mu(t)g(t)dt + C}{\mu(t)} = \frac{\int t^2(4t)dt + C}{t^2} = \frac{1}{t^2}\left[\int 4t^3 dt + C\right] = t^2 + \frac{C}{t^2}$$

$$ty' + 2y = 4t^2, \quad y(1) = 2,$$

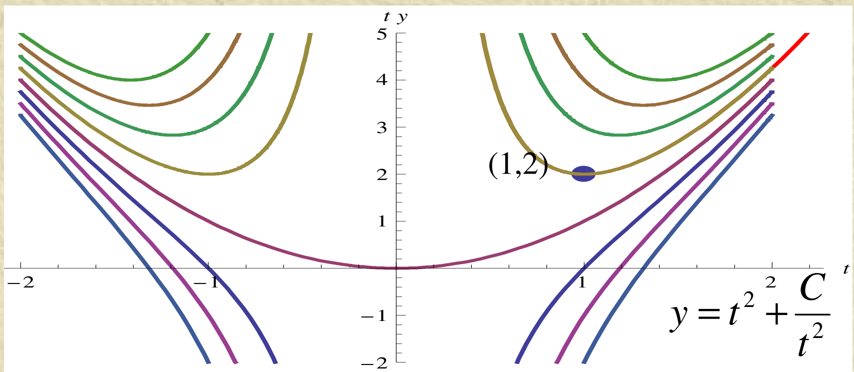
Example 3: Particular Solution (2 of 2)

- Using the initial condition $y(1) = 2$ and general solution

$$y = t^2 + \frac{C}{t^2}, y(1) = 1 + C = 2 \Rightarrow C = 1$$

it follows that $$y = t^2 + \frac{1}{t^2}$$

- The graphs below show solution curves for the differential equation, including a particular solution whose graph contains the initial point (1,2). Notice that when C=0, we get the parabolic solution $y = t^2$ (shown) and that solution separates the solutions into those that are asymptotic to the positive versus negative y-axis.



Example 4: A Solution in Integral Form (1 of 2)

- To solve the initial value problem

$$2y' + ty = 2, \quad y(0) = 1,$$

first put into standard form:

$$y' + \frac{t}{2}y = 1$$

- Then

$$\mu(t) = e^{\int p(t)dt} = e^{\int \frac{t}{2}dt} = e^{\frac{t^2}{4}}$$

and hence

$$y = e^{-t^2/4}\left(\int_0^t e^{s^2/4}ds + C\right) = e^{-t^2/4}\left(\int_0^t e^{s^2/4}ds\right) + Ce^{-t^2/4}$$

$$2y' + ty = 2, \quad y(0) = 1,$$

Example 4: A Solution in Integral Form (2 of 2)

- Notice that this solution must be left in the form of an integral, since there is no closed form for the integral.

$$y = e^{-t^2/4}\left(\int_0^t e^{s^2/4} ds\right) + Ce^{-t^2/4}$$

- Using software such as *Mathematica* or Maple, we can approximate the solution for the given initial conditions as well as for other initial conditions.
- Several solution curves are shown.