Boyce/DiPrima 9th ed, Ch 3.1: 2nd Order Linear Homogeneous Equations-Constant Coefficients

Elementary Differential Equations and Boundary Value Problems, 9th edition, by William E. Boyce and Richard C. DiPrima, ©2009 by John Wiley & Sons, Inc.

- A **second order ordinary differential equation** has the general form

$$y'' = f(t, y, y')$$

where f is some given function.

- This equation is said to be **linear** if f is linear in y and y':

$$y'' = g(t) - p(t)y' - q(t)y$$

Otherwise the equation is said to be **nonlinear**.

- A second order linear equation often appears as

$$P(t)y'' + Q(t)y' + R(t)y = G(t)$$

- If $G(t) = 0$ for all t, then the equation is called **homogeneous**. Otherwise the equation is **nonhomogeneous**.

Homogeneous Equations, Initial Values

- In Sections 3.5 and 3.6, we will see that once a solution to a homogeneous equation is found, then it is possible to solve the corresponding nonhomogeneous equation, or at least express the solution in terms of an integral.
- The focus of this chapter is thus on homogeneous equations; and in particular, those with constant coefficients:

  $$ay'' + by' + cy = 0$$

  We will examine the variable coefficient case in Chapter 5.
- Initial conditions typically take the form

  $$y(t_0) = y_0, \;\; y'(t_0) = y_0'$$

- Thus solution passes through (t_0, y_0), and slope of solution at (t_0, y_0) is equal to y_0'.

Example 1: Infinitely Many Solutions

- Consider the second order linear differential equation

$$y'' - y = 0$$

- Two solutions of this equation are

$$y_1(t) = e^t, \quad y_2(t) = e^{-t}$$

- Other solutions include

$$y_3(t) = 3e^t, \quad y_4(t) = 5e^{-t}, \quad y_5(t) = 3e^t + 5e^{-t}$$

- Based on these observations, we see that there are infinitely many solutions of the form

$$y(t) = c_1 e^t + c_2 e^{-t}$$

- It will be shown in Section 3.2 that all solutions of the differential equation above can be expressed in this form.

Example 1: Initial Conditions (2 of 3)

- Now consider the following initial value problem for our equation:

$$y'' - y = 0, \quad y(0) = 2, \; y'(0) = -1$$

- We have found a general solution of the form

$$y(t) = c_1 e^t + c_2 e^{-t}$$

- Using the initial equations,

$$\left.\begin{aligned} y(0) &= c_1 + c_2 = 2 \\ y'(0) &= c_1 - c_2 = -1 \end{aligned}\right\} \Rightarrow c_1 = 1/2, \; c_2 = 3/2$$

- Thus

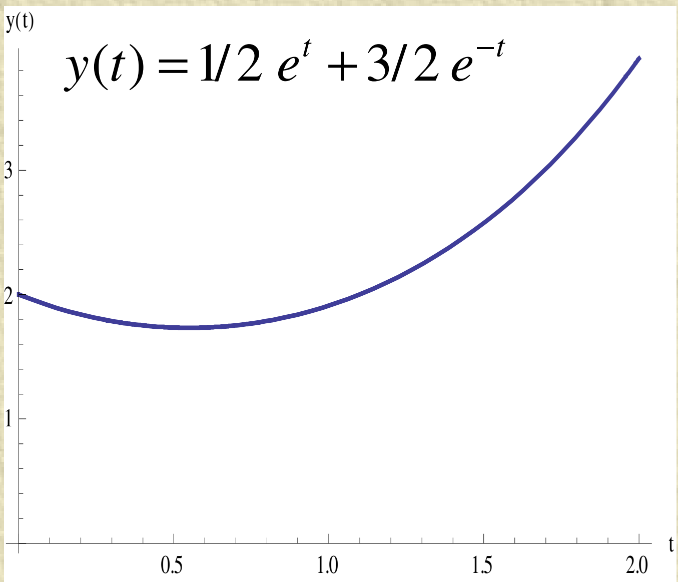
$$y(t) = 1/2 \; e^t + 3/2 \; e^{-t}$$

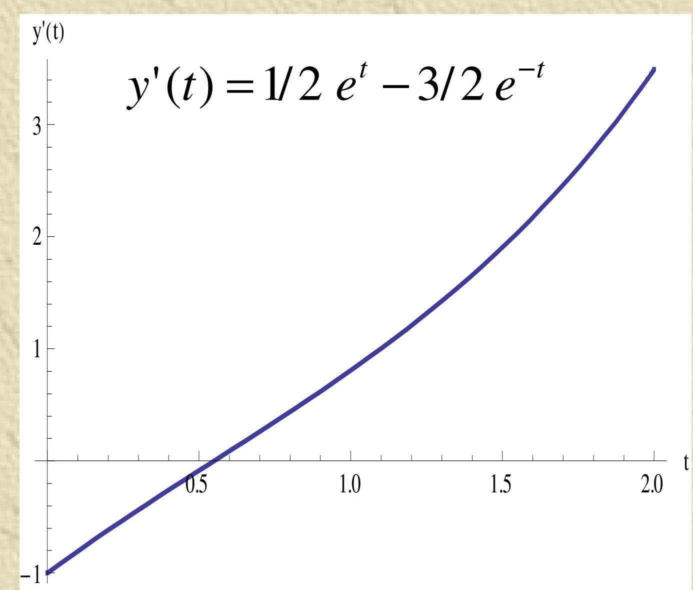
Example 1: Solution Graphs (3 of 3)

- Our initial value problem and solution are

$$y'' - y = 0, \quad y(0) = 2,\ y'(0) = -1 \ \Rightarrow \ y(t) = 1/2\, e^{t} + 3/2\, e^{-t}$$

- Graphs of both y(t) and y′(t) are given below. Observe that both initial conditions are satisfied.





Characteristic Equation

- To solve the 2nd order equation with constant coefficients,

$$ay'' + by' + cy = 0,$$

  we begin by assuming a solution of the form $y = e^{rt}$.
- Substituting this into the differential equation, we obtain

$$ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0$$

- Simplifying,

$$e^{rt}(ar^2 + br + c) = 0$$

  and hence

$$ar^2 + br + c = 0$$

- This last equation is called the **characteristic equation** of the differential equation.
- We then solve for r by factoring or using quadratic formula.

General Solution

- Using the quadratic formula on the characteristic equation

$$ar^2 + br + c = 0,$$

  we obtain two solutions, r_1 and r_2.
- There are three possible results:
  - The roots r_1, r_2 are real and $r_1 \neq r_2$.
  - The roots r_1, r_2 are real and $r_1 = r_2$.
  - The roots r_1, r_2 are complex.

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- In this section, we will assume r_1, r_2 are real and $r_1 \neq r_2$.
- In this case, the general solution has the form

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Initial Conditions

- For the initial value problem

$$ay'' + by' + cy = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_0',$$

  we use the general solution

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

  together with the initial conditions to find c_1 and c_2. That is,

$$\left.\begin{aligned} c_1 e^{r_1 t_0} + c_2 e^{r_2 t_0} &= y_0 \\ c_1 r_1 e^{r_1 t_0} + c_2 r_2 e^{r_2 t_0} &= y_0' \end{aligned}\right\} \Rightarrow c_1 = \frac{y_0' - y_0 r_2}{r_1 - r_2} e^{-r_1 t_0}, \; c_2 = \frac{y_0 r_1 - y_0'}{r_1 - r_2} e^{-r_2 t_0}$$

- Since we are assuming $r_1 \neq r_2$, it follows that a solution of the form $y = e^{rt}$ to the above initial value problem will always exist, for any set of initial conditions.

Example 2 (General Solution)

- Consider the linear differential equation

$$y'' + 5y' + 6y = 0$$

- Assuming an exponential solution leads to the characteristic equation:

$$y(t) = e^{rt} \quad \Rightarrow \quad r^2 + 5r + 6 = 0 \iff (r+2)(r+3) = 0$$

- Factoring the characteristic equation yields two solutions: $r_1 = -2$ and $r_2 = -3$
- Therefore, the general solution to this differential equation has the form

$$y(t) = c_1 e^{-2t} + c_2 e^{-3t}$$

Example 3 (Particular Solution)

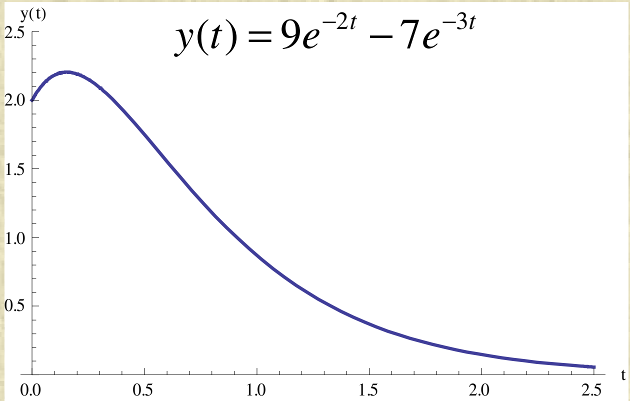
- Consider the initial value problem

$$y'' + 5y' + 6y = 0, \quad y(0) = 2, \quad y'(0) = 3$$

- From the preceding example, we know the general solution has the form: $y(t) = c_1 e^{-2t} + c_2 e^{-3t}$
- With derivative: $y'(t) = -2c_1 e^{-2t} - 3c_2 e^{-3t}$
- Using the initial conditions:

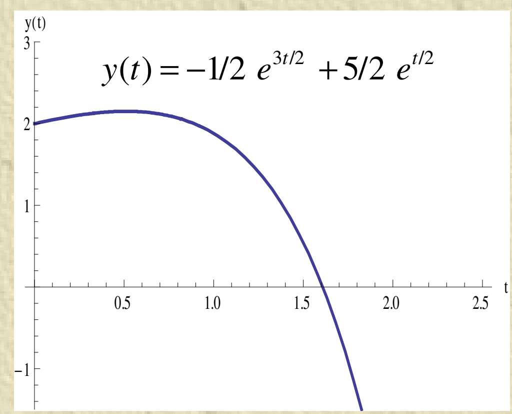
$$\left.\begin{aligned} c_1 + c_2 &= 1 \\ -2c_1 - 3c_2 &= 3 \end{aligned}\right\} \Rightarrow c_1 = 9,\ c_2 = -7$$

- Thus $y(t) = 9e^{-2t} - 7e^{-3t}$



Example 4: Initial Value Problem

- Consider the initial value problem
  $$4y'' - 8y' + 3y = 0, \quad y(0) = 2, \;\; y'(0) = 1/2$$
- Then
  $$y(t) = e^{rt} \quad \Rightarrow \quad 4r^2 - 8r + 3 = 0 \;\Leftrightarrow\; (2r-3)(2r-1) = 0$$
- Factoring yields two solutions, $r_1 = 3/2$ and $r_2 = 1/2$
- The general solution has the form
  $$y(t) = c_1 e^{3t/2} + c_2 e^{t/2}$$
- Using initial conditions:
  $$\left.\begin{aligned} c_1 + c_2 &= 2 \\ 3/2\,c_1 + 1/2\,c_2 &= 1/2 \end{aligned}\right\} \Rightarrow c_1 = -1/2 \,,\; c_2 = 5/2$$
- Thus $\quad y(t) = -1/2\, e^{3t/2} + 5/2\, e^{t/2}$



Example 5: Find Maximum Value

- For the initial value problem in Example 3, to find the maximum value attained by the solution, we set $y'(t) = 0$ and solve for t:

$$y(t) = 9e^{-2t} - 7e^{-3t}$$

$$y'(t) = -18e^{-2t} + 21e^{-3t} \overset{set}{=} 0$$

$$6e^{-2t} = 7e^{-3t}$$

$$e^{t} = 7/6$$

$$t = \ln(7/6)$$

$$t \approx 0.1542$$

$$y \approx 2.204$$