Boyce/DiPrima 9th ed, Ch 7.7: Fundamental Matrices

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- Suppose that $\mathbf{x}^{(1)}(t),\ldots, \mathbf{x}^{(n)}(t)$ form a fundamental set of solutions for $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on $\alpha < t < \beta$.
- The matrix

$$\mathbf{\Psi}(t) = \begin{pmatrix} x_1^{(1)}(t) & \cdots & x_1^{(n)}(t) \\ \vdots & \ddots & \vdots \\ x_n^{(1)}(t) & \cdots & x_n^{(n)}(t) \end{pmatrix},$$

  whose columns are $\mathbf{x}^{(1)}(t),\ldots, \mathbf{x}^{(n)}(t)$, is a fundamental matrix for the system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$. This matrix is nonsingular since its columns are linearly independent, and hence $\det\mathbf{\Psi} \neq 0$.
- Note also that since $\mathbf{x}^{(1)}(t),\ldots, \mathbf{x}^{(n)}(t)$ are solutions of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$, $\mathbf{\Psi}$ satisfies the matrix differential equation $\mathbf{\Psi}' = \mathbf{P}(t)\mathbf{\Psi}$.

Example 1:

- Consider the homogeneous equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$ below.

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}$$

- In Chapter 7.5, we found the following fundamental solutions for this system:

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}, \ \ \mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$$

- Thus a fundamental matrix for this system is

$$\mathbf{\Psi}(t) = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}$$

Fundamental Matrices and General Solution

- The general solution of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$

$$\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}$$

can be expressed $\mathbf{x} = \mathbf{\Psi}(t)\mathbf{c}$, where $\mathbf{c}$ is a constant vector with components $c_1, \ldots, c_n$:

$$\mathbf{x} = \mathbf{\Psi}(t)\mathbf{c} = \begin{pmatrix} x_1^{(1)}(t) & \cdots & x_1^{(n)}(t) \\ \vdots & \ddots & \vdots \\ x_n^{(1)}(t) & \cdots & x_n^{(n)}(t) \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

Fundamental Matrix & Initial Value Problem

- Consider an initial value problem

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \quad \mathbf{x}(t_0) = \mathbf{x}^0$$

where $\alpha < t_0 < \beta$ and $\mathbf{x}^0$ is a given initial vector.

- Now the solution has the form $\mathbf{x} = \mathbf{\Psi}(t)\mathbf{c}$, hence we choose $\mathbf{c}$ so as to satisfy $\mathbf{x}(t_0) = \mathbf{x}^0$.
- Recalling $\mathbf{\Psi}(t_0)$ is nonsingular, it follows that

$$\mathbf{\Psi}(t_0)\mathbf{c} = \mathbf{x}^0 \quad \Rightarrow \quad \mathbf{c} = \mathbf{\Psi}^{-1}(t_0)\mathbf{x}^0$$

- Thus our solution $\mathbf{x} = \mathbf{\Psi}(t)\mathbf{c}$ can be expressed as

$$\mathbf{x} = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0)\mathbf{x}^0$$

Recall: Theorem 7.4.4

- Let

$$\mathbf{e}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ \mathbf{e}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \ \mathbf{e}^{(n)} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

- Let $\mathbf{x}^{(1)},\ldots, \mathbf{x}^{(n)}$ be solutions of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on I: $\alpha < t < \beta$ that satisfy the initial conditions

$$\mathbf{x}^{(1)}(t_0) = \mathbf{e}^{(1)}, \ \ldots, \ \mathbf{x}^{(n)}(t_0) = \mathbf{e}^{(n)}, \ \alpha < t_0 < \beta$$

  Then $\mathbf{x}^{(1)},\ldots, \mathbf{x}^{(n)}$ are fundamental solutions of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$.

Fundamental Matrix & Theorem 7.4.4

- Suppose $\mathbf{x}^{(1)}(t),\ldots, \mathbf{x}^{(n)}(t)$ form the fundamental solutions given by Thm 7.4.4. Denote the corresponding fundamental matrix by $\mathbf{\Phi}(t)$. Then columns of $\mathbf{\Phi}(t)$ are $\mathbf{x}^{(1)}(t),\ldots, \mathbf{x}^{(n)}(t)$, and hence

$$\Phi(t_0) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = \mathbf{I}$$

- Thus $\mathbf{\Phi}^{-1}(t_0) = \mathbf{I}$, and the hence general solution to the corresponding initial value problem is

$$\mathbf{x} = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)\mathbf{x}^0 = \mathbf{\Phi}(t)\mathbf{x}^0$$

- It follows that for any fundamental matrix $\mathbf{\Psi}(t)$,

$$\mathbf{x} = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0)\mathbf{x}^0 = \mathbf{\Phi}(t)\mathbf{x}^0 \;\Rightarrow\; \mathbf{\Phi}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0)$$

The Fundamental Matrix $\mathbf{\Phi}$ and Varying Initial Conditions

- Thus when using the fundamental matrix $\mathbf{\Phi}(t)$, the general solution to an IVP is

$$\mathbf{x} = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)\mathbf{x}^0 = \mathbf{\Phi}(t)\mathbf{x}^0$$

- This representation is useful if same system is to be solved for many different initial conditions, such as a physical system that can be started from many different initial states.
- Also, once $\mathbf{\Phi}(t)$ has been determined, the solution to each set of initial conditions can be found by matrix multiplication, as indicated by the equation above.
- Thus $\mathbf{\Phi}(t)$ represents a linear transformation of the initial conditions $\mathbf{x}^0$ into the solution $\mathbf{x}(t)$ at time t.

Example 2: Find $\Phi(t)$ for 2 x 2 System (1 of 5)

- Find $\Phi(t)$ such that $\Phi(0) = \mathbf{I}$ for the system below.

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}$$

- Solution: First, we must obtain $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ such that

$$\mathbf{x}^{(1)}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \mathbf{x}^{(2)}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- We know from previous results that the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$$

- Every solution can be expressed in terms of the general solution, and we use this fact to find $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$.

Example 2: Use General Solution

- Thus, to find $\mathbf{x}^{(1)}(t)$, express it terms of the general solution

$$\mathbf{x}^{(1)}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$$

  and then find the coefficients c_1 and c_2.

- To do so, use the initial conditions to obtain

$$\mathbf{x}^{(1)}(0) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

  or equivalently,

$$\begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Example 2: Solve for $\mathbf{x}^{(1)}(t)$ (3 of 5)

- To find $\mathbf{x}^{(1)}(t)$, we therefore solve

$$\begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

by row reducing the augmented matrix:

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & -4 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1/2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \end{pmatrix}$$

$$\rightarrow \begin{array}{lcl} c_1 & & = 1/2 \\ & c_2 & = 1/2 \end{array}$$

- Thus

$$\mathbf{x}^{(1)}(t) = \frac{1}{2}\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + \frac{1}{2}\begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} = \begin{pmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \\ e^{3t} - e^{-t} \end{pmatrix}$$

Example 2: Solve for $\mathbf{x}^{(2)}(t)$ (4 of 5)

- To find $\mathbf{x}^{(2)}(t)$, we similarly solve

$$\begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

by row reducing the augmented matrix:

$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & -4 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1/4 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1/4 \\ 0 & 1 & -1/4 \end{pmatrix}$$

$$\to \begin{array}{ll} c_1 & = 1/4 \\ c_2 & = -1/4 \end{array}$$

- Thus

$$\mathbf{x}^{(2)}(t) = \frac{1}{4}\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} - \frac{1}{4}\begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} = \begin{pmatrix} \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} \\ \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \end{pmatrix}$$

Example 2: Obtain $\mathbf{\Phi}(t)$ (5 of 5)

- The columns of $\mathbf{\Phi}(t)$ are given by $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$, and thus from the previous slide we have

$$\mathbf{\Phi}(t) = \begin{pmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} & \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} \\ e^{3t} - e^{-t} & \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \end{pmatrix}$$

- Note $\mathbf{\Phi}(t)$ is more complicated than $\mathbf{\Psi}(t)$ found in Ex 1. However, now that we have $\mathbf{\Phi}(t)$, it is much easier to determine the solution to any set of initial conditions.

$$\mathbf{\Psi}(t) = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}$$

Matrix Exponential Functions

- Consider the following two cases:
  - The solution to $x' = ax$, $x(0) = x_0$, is $x = x_0 e^{at}$, where $e^0 = 1$.
  - The solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}^0$, is $\mathbf{x} = \mathbf{\Phi}(t)\mathbf{x}^0$, where $\mathbf{\Phi}(0) = \mathbf{I}$.
- Comparing the form and solution for both of these cases, we might expect $\mathbf{\Phi}(t)$ to have an exponential character.
- Indeed, it can be shown that $\mathbf{\Phi}(t) = e^{\mathbf{A}t}$, where

$$e^{\mathbf{A}t} = \sum_{n=0}^{\infty} \frac{\mathbf{A}^n t^n}{n!} = \mathbf{I} + \sum_{n=1}^{\infty} \frac{\mathbf{A}^n t^n}{n!}$$

  is a well defined matrix function that has all the usual properties of an exponential function. See text for details.
- Thus the solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}^0$, is $\mathbf{x} = e^{\mathbf{A}t}\mathbf{x}^0$.

Coupled Systems of Equations

- Recall that our constant coefficient homogeneous system

$$
\begin{aligned}
x_1' &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\
&\vdots \\
x_n' &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n,
\end{aligned}
$$

written as $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with

$$
\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix},
$$

is a system of *coupled* equations that must be solved *simultaneously* to find all the unknown variables.

Uncoupled Systems & Diagonal Matrices

- In contrast, if each equation had only one variable, solved for independently of other equations, then task would be easier.
- In this case our system would have the form

$$
\begin{aligned}
x_1' &= d_{11}x_1 + 0x_2 + \ldots + 0x_n \\
x_2' &= 0x_1 + d_{11}x_2 + \ldots + 0x_n \\
&\vdots \\
x_n' &= 0x_1 + 0x_2 + \ldots + d_{nn}x_n,
\end{aligned}
$$

or $\mathbf{x}' = \mathbf{D}\mathbf{x}$, where $\mathbf{D}$ is a diagonal matrix:

$$
\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{pmatrix}
$$

Uncoupling: Transform Matrix T

- In order to explore transforming our given system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ of coupled equations into an uncoupled system $\mathbf{x}' = \mathbf{D}\mathbf{x}$, where $\mathbf{D}$ is a diagonal matrix, we will use the eigenvectors of $\mathbf{A}$.
- Suppose $\mathbf{A}$ is n x n with n linearly independent eigenvectors $\boldsymbol{\xi}^{(1)},\ldots, \boldsymbol{\xi}^{(n)}$, and corresponding eigenvalues $\lambda_1,\ldots, \lambda_n$.
- Define n x n matrices $\mathbf{T}$ and $\mathbf{D}$ using the eigenvalues & eigenvectors of $\mathbf{A}$:

$$\mathbf{T} = \begin{pmatrix} \xi_1^{(1)} & \cdots & \xi_1^{(n)} \\ \vdots & \ddots & \vdots \\ \xi_n^{(1)} & \cdots & \xi_n^{(n)} \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

- Note that $\mathbf{T}$ is nonsingular, and hence $\mathbf{T}^{-1}$ exists.

Uncoupling: $\mathbf{T}^{-1}\mathbf{AT} = \mathbf{D}$

- Recall here the definitions of **A**, **T** and **D**:

$$\mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}, \quad \mathbf{T} = \begin{pmatrix} \xi_1^{(1)} & \cdots & \xi_1^{(n)} \\ \vdots & \ddots & \vdots \\ \xi_n^{(1)} & \cdots & \xi_n^{(n)} \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

- Then the columns of **AT** are $\mathbf{A}\boldsymbol{\xi}^{(1)}, \ldots, \mathbf{A}\boldsymbol{\xi}^{(n)}$, and hence

$$\mathbf{AT} = \begin{pmatrix} \lambda_1 \xi_1^{(1)} & \cdots & \lambda_n \xi_1^{(n)} \\ \vdots & \ddots & \vdots \\ \lambda_1 \xi_n^{(1)} & \cdots & \lambda_n \xi_n^{(n)} \end{pmatrix} = \mathbf{TD}$$

- It follows that $\mathbf{T}^{-1}\mathbf{AT} = \mathbf{D}$.

Similarity Transformations

- Thus, if the eigenvalues and eigenvectors of **A** are known, then **A** can be transformed into a diagonal matrix **D**, with

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{D}$$

- This process is known as a **similarity transformation**, and **A** is said to be **similar** to **D**. Alternatively, we could say that **A** is **diagonalizable**.

$$\mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}, \quad \mathbf{T} = \begin{pmatrix} \xi_1^{(1)} & \cdots & \xi_1^{(n)} \\ \vdots & \ddots & \vdots \\ \xi_n^{(1)} & \cdots & \xi_n^{(n)} \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

Similarity Transformations: Hermitian Case

- Recall: Our similarity transformation of **A** has the form

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{D}$$

  where **D** is diagonal and columns of **T** are eigenvectors of **A**.
- If **A** is Hermitian, then **A** has n linearly independent orthogonal eigenvectors $\xi^{(1)},\ldots,\xi^{(n)}$, normalized so that $(\xi^{(i)}, \xi^{(i)}) = 1$ for $i = 1,\ldots, n$, and $(\xi^{(i)}, \xi^{(k)}) = 0$ for $i \neq k$.
- With this selection of eigenvectors, it can be shown that $\mathbf{T}^{-1} = \mathbf{T}^*$. In this case we can write our similarity transform as

$$\mathbf{T}^*\mathbf{A}\mathbf{T} = \mathbf{D}$$

Nondiagonalizable A

- Finally, if **A** is n x n with fewer than n linearly independent eigenvectors, then there is no matrix **T** such that $\mathbf{T}^{-1}\mathbf{AT} = \mathbf{D}$.
- In this case, **A** is not similar to a diagonal matrix and **A** is not diagonlizable.

$$\mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}, \quad \mathbf{T} = \begin{pmatrix} \xi_1^{(1)} & \cdots & \xi_1^{(n)} \\ \vdots & \ddots & \vdots \\ \xi_n^{(1)} & \cdots & \xi_n^{(n)} \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

Example 3: Find Transformation Matrix T (1 of 2)

- For the matrix **A** below, find the similarity transformation matrix **T** and show that **A** can be diagonalized.

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$$

- We already know that the eigenvalues are $\lambda_1 = 3$, $\lambda_2 = -1$ with corresponding eigenvectors

$$\boldsymbol{\xi}^{(1)}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \ \boldsymbol{\xi}^{(2)}(t) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

- Thus

$$\mathbf{T} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}, \ \mathbf{D} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

Example 3: Similarity Transformation

- To find $\mathbf{T}^{-1}$, augment the identity to $\mathbf{T}$ and row reduce:

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & -2 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & -4 & -2 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1/2 & -1/4 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 1/2 & 1/4 \\ 0 & 1 & 1/2 & -1/4 \end{pmatrix} \rightarrow \mathbf{T}^{-1} = \begin{pmatrix} 1/2 & 1/4 \\ 1/2 & -1/4 \end{pmatrix}$$

- Then

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{pmatrix} 1/2 & 1/4 \\ 1/2 & -1/4 \end{pmatrix} \left[\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \right]$$

$$= \begin{pmatrix} 1/2 & 1/4 \\ 1/2 & -1/4 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 6 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} = \mathbf{D}$$

- Thus **A** is similar to **D**, and hence **A** is diagonalizable.

Fundamental Matrices for Similar Systems (1 of 3)

- Recall our original system of differential equations $\mathbf{x}' = \mathbf{A}\mathbf{x}$.
- If $\mathbf{A}$ is n x n with n linearly independent eigenvectors, then $\mathbf{A}$ is diagonalizable. The eigenvectors form the columns of the nonsingular transform matrix $\mathbf{T}$, and the eigenvalues are the corresponding nonzero entries in the diagonal matrix $\mathbf{D}$.
- Suppose $\mathbf{x}$ satisfies $\mathbf{x}' = \mathbf{A}\mathbf{x}$, let $\mathbf{y}$ be the n x 1 vector such that $\mathbf{x} = \mathbf{T}\mathbf{y}$. That is, let $\mathbf{y}$ be defined by $\mathbf{y} = \mathbf{T}^{-1}\mathbf{x}$.
- Since $\mathbf{x}' = \mathbf{A}\mathbf{x}$ and $\mathbf{T}$ is a constant matrix, we have $\mathbf{T}\mathbf{y}' = \mathbf{A}\mathbf{T}\mathbf{y}$, and hence $\mathbf{y}' = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\mathbf{y} = \mathbf{D}\mathbf{y}$.
- Therefore $\mathbf{y}$ satisfies $\mathbf{y}' = \mathbf{D}\mathbf{y}$, the system similar to $\mathbf{x}' = \mathbf{A}\mathbf{x}$.
- Both of these systems have fundamental matrices, which we examine next.

Fundamental Matrix for Diagonal System (2 of 3)

- A fundamental matrix for $\mathbf{y}' = \mathbf{D}\mathbf{y}$ is given by $\mathbf{Q}(t) = e^{\mathbf{D}t}$.
- Recalling the definition of $e^{\mathbf{D}t}$, we have

$$\mathbf{Q}(t) = \sum_{n=0}^{\infty} \frac{\mathbf{D}^n t^n}{n!} = \sum_{n=0}^{\infty} \begin{pmatrix} \lambda_1^n & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n^n \end{pmatrix} \frac{t^n}{n!} = \begin{pmatrix} \sum_{n=0}^{\infty} \frac{(\lambda_1 t)^n}{n!} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sum_{n=0}^{\infty} \frac{(\lambda_n t)^n}{n!} \end{pmatrix}$$

$$= \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e^{\lambda_n t} \end{pmatrix}$$

Fundamental Matrix for Original System (3 of 3)

- To obtain a fundamental matrix $\boldsymbol{\Psi}(t)$ for $\mathbf{x}' = \mathbf{A}\mathbf{x}$, recall that the columns of $\boldsymbol{\Psi}(t)$ consist of fundamental solutions $\mathbf{x}$ satisfying $\mathbf{x}' = \mathbf{A}\mathbf{x}$. We also know $\mathbf{x} = \mathbf{T}\mathbf{y}$, and hence it follows that

$$\boldsymbol{\Psi} = \mathbf{TQ} = \begin{pmatrix} \xi_1^{(1)} & \cdots & \xi_1^{(n)} \\ \vdots & \ddots & \vdots \\ \xi_n^{(1)} & \cdots & \xi_n^{(n)} \end{pmatrix} \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e^{\lambda_n t} \end{pmatrix} = \begin{pmatrix} \xi_1^{(1)} e^{\lambda_1 t} & \cdots & \xi_1^{(n)} e^{\lambda_n t} \\ \vdots & \ddots & \vdots \\ \xi_n^{(1)} e^{\lambda_1 t} & \cdots & \xi_n^{(n)} e^{\lambda_n t} \end{pmatrix}$$

- The columns of $\boldsymbol{\Psi}(t)$ given the expected fundamental solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Example 4:
Fundamental Matrices for Similar Systems

- We now use the analysis and results of the last few slides.
- Applying the transformation $\mathbf{x} = \mathbf{T}\mathbf{y}$ to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ below, this system becomes $\mathbf{y}' = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\mathbf{y} = \mathbf{D}\mathbf{y}$:

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\mathbf{x} \quad \Rightarrow \quad \mathbf{y}' = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{y}$$

- A fundamental matrix for $\mathbf{y}' = \mathbf{D}\mathbf{y}$ is given by $\mathbf{Q}(t) = e^{\mathbf{D}t}$:

$$\mathbf{Q}(t) = \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix}$$

- Thus a fundamental matrix $\mathbf{\Psi}(t)$ for $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is

$$\mathbf{\Psi}(t) = \mathbf{T}\mathbf{Q} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix} = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}$$