Lecture 10
MATH-42021/52021 Graph Theory and Combinatorics.

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Trees - basic definitions

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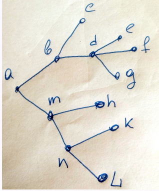

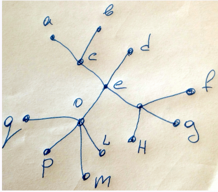

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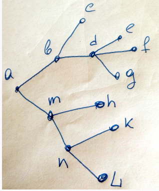
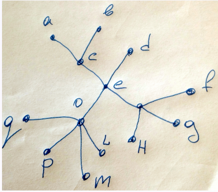
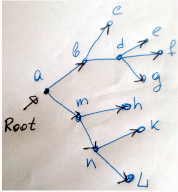
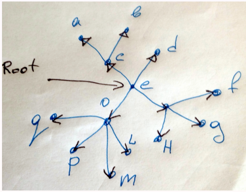




Another way to define a tree as a graph with special designated vertex - *root* such that there is a unique pass from a root to any other vertex in the tree. One can also use this definition for directed graphs.





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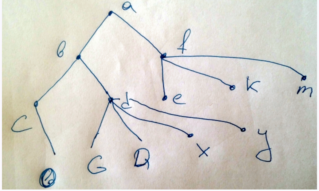

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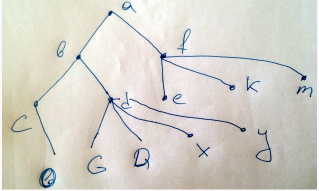

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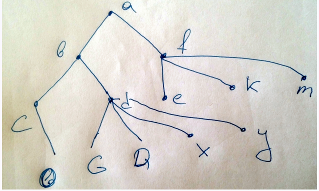

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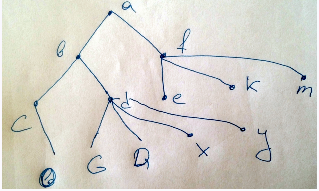

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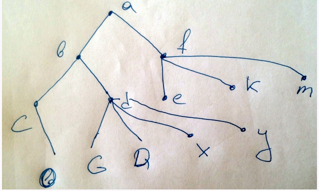

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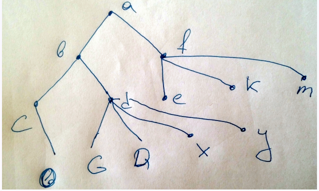

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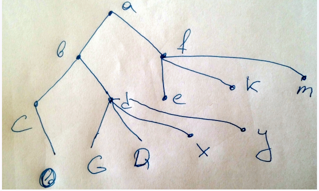

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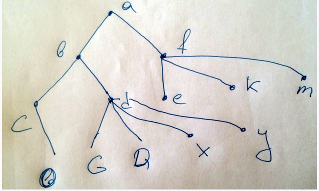

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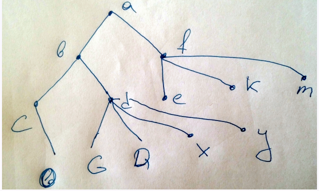

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Vertices of T with no children are called *leaves* of T (O, G, Q, x, y in our example).
Vertices with children are called *internal* vertices.

Theorem.

A tree with n vertices has $n-1$ edges.

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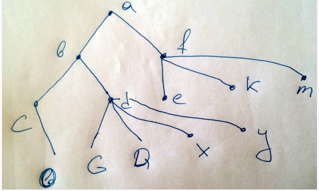
Proof : We may assume that the tree is rooted (we select the root and structure the tree as before).



Then we can pair every vertex (but the root!) with unique incoming edge from its parent. This gives exactly $n-1$ edges.

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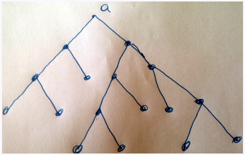

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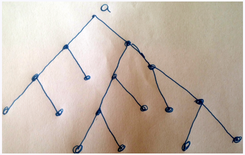

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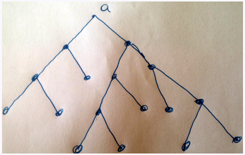

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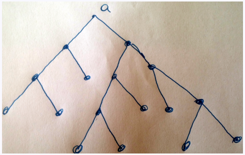

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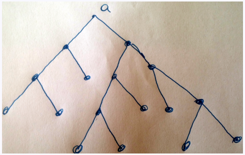

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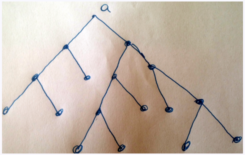

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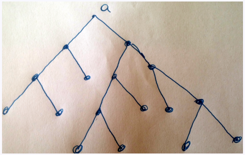

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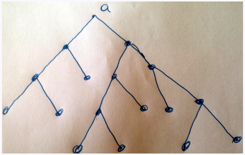

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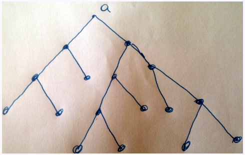

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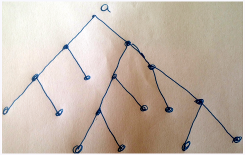

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The proof follows immediately from the theorem and the fact that $l + i = n$.

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$i=(l-1)/(m-1)=47/1=47$.

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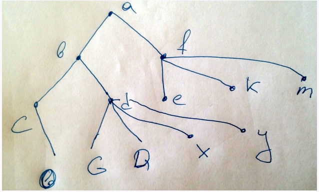

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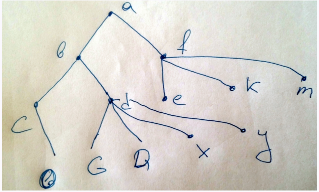

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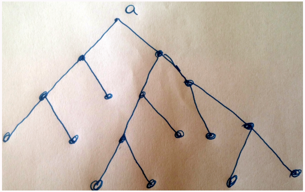

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The idea is to form a tree. The approach that we can always divide the coins into three almost equal piles (for example round $n/3$ and make one pile to contain the reminder). Compare two piles with the same number of coins. If the weight equal - bad coin is in the third pile. If one is lighter there your bad coin.

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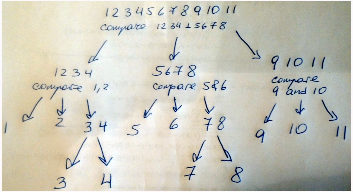

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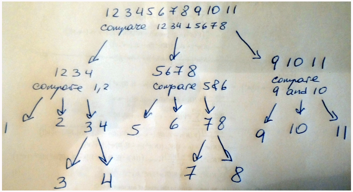

This way we create a ternary tree with n leaves, thus the hight must be $\lceil \log_3 n \rceil$.

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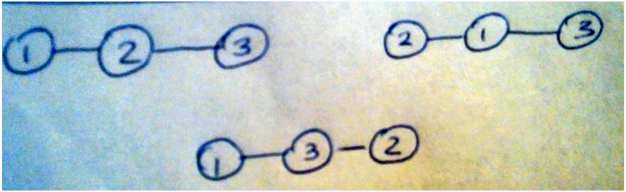

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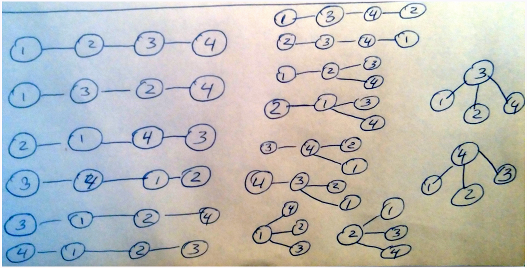

How many (undirected) trees are there – general theorem.

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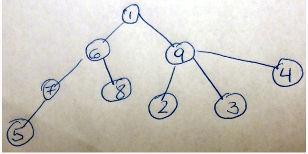

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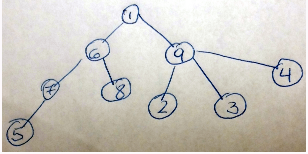

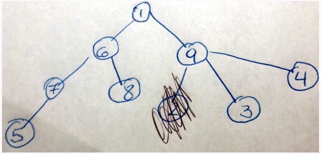

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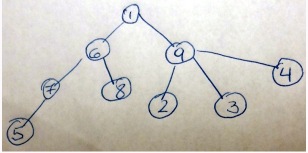

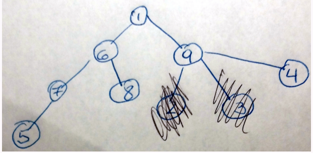

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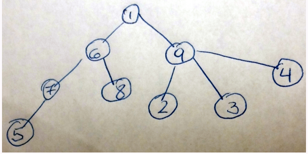

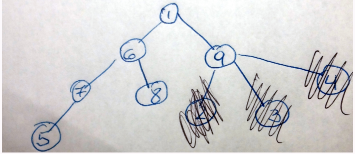

Now we remove the vertex l_1 and repeat the process. For our example we will get $l_2 = 3$ and $s_2 = 9$; $l_3 = 4$ and $s_3 = 9$;

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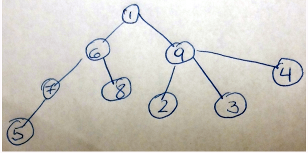

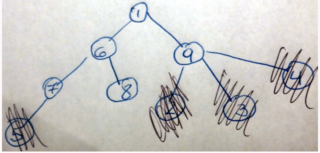

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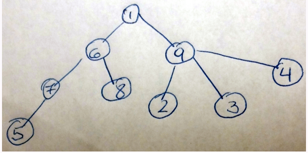

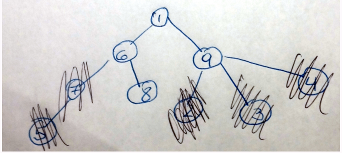

Now we remove the vertex l_1 and repeat the process. For our example we will get $l_2 = 3$ and $s_2 = 9$; $l_3 = 4$ and $s_3 = 9$; $l_4 = 5$ and $s_4 = 7$; $l_5 = 7$ and $s_5 = 6$;

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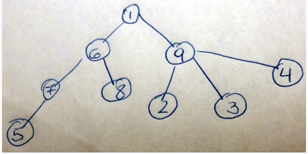

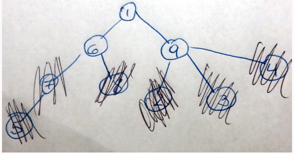

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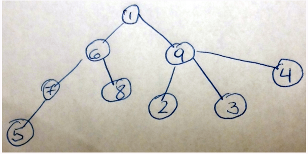

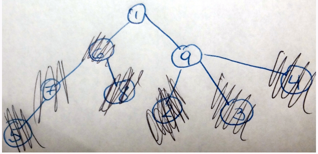

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How many (undirected) trees are there – general theorem.

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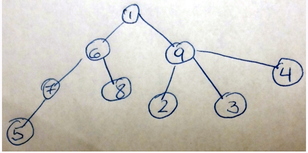

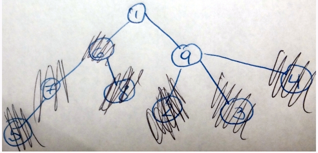

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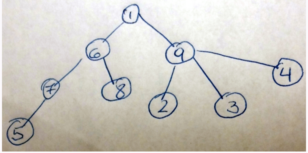

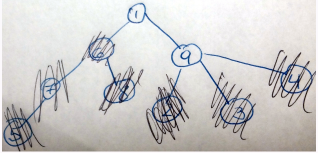

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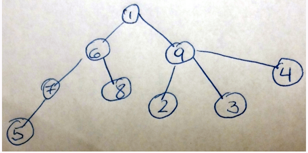

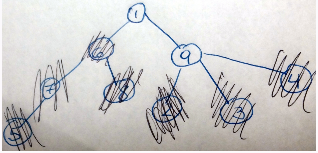

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Our next goal is to show that any such $(n-2)$-length sequence defines a unique n-item tree.

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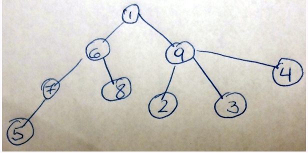

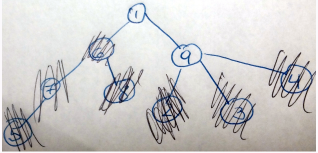

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How many (undirected) trees are there – general theorem.

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There are n^{n-2} different undirected trees on n items.

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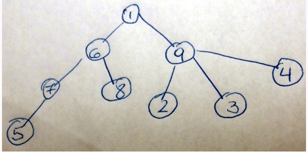

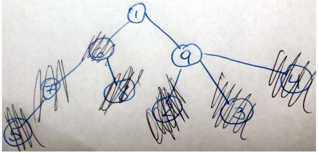

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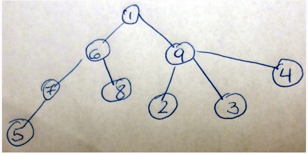
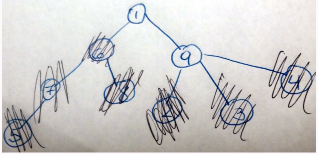
Theorem

There are n^{n-2} different undirected trees on n items.

Proof : First notice that n^{n-2} is a number of different sequences of length $(n-2)$ where each element is selected out of n (notice that we do not ask elements of the sequence to be different, thus we can select the first element in n ways, after the second element in n ways and continue till $(n-2)$'nd element, we also note that the order does meter for sequences).
Also note, that we now can prove the theorem by creating a one to one correspondence between trees of n-elements and sequences of n-elements. For any tree of n elements we will create a sequence $(s_1, \dots, s_{n-2})$ of length $n-2$ as follows. Let l_1 be a leaf of the tree with smallest number and let s_1 be the number of the one vertex adjacent to it. For example on the tree below l_1 is a leaf with number 2 on it and $s_1 = 9$.





Now we remove the vertex l_1 and repeat the process. For our example we will get $l_2 = 3$ and $s_2 = 9$; $l_3 = 4$ and $s_3 = 9$; $l_4 = 5$ and $s_4 = 7$; $l_5 = 7$ and $s_5 = 6$; $l_6 = 8$ and $s_6 = 6$; $l_7 = 9$ and $s_7 = 1$. We got sequence $(9, 9, 9, 7, 6, 6, 1)$ corresponding to our graph. Such sequence are called *Prufer Sequences.*
Our next goal is to show that any such $(n-2)$-length sequence defines a unique n-item tree. To do this we simply reverse the process. Observe that leaves (vertices of degree 1) will never appear in the sequence. So we draw them first.

How many (undirected) trees are there – general theorem.

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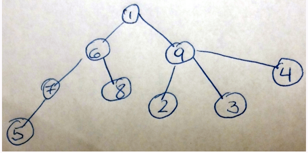

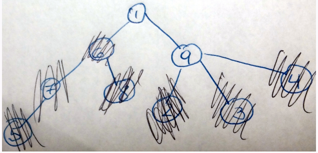

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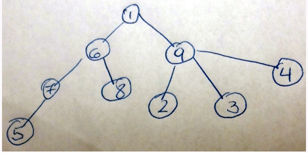

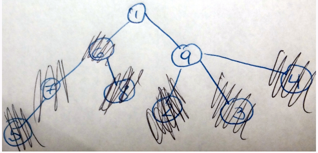

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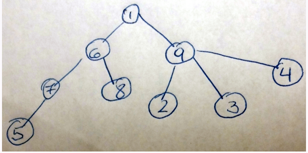

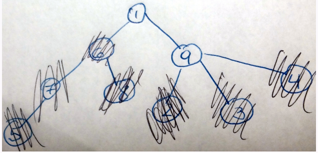

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