# Lecture 11
# MATH-42021/52021 Graph Theory and Combinatorics.

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Two different principles:

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## The Addition Principle

Assume there are $n_1$ **different** objects in the first set, $n_2$ **different** objects in the second set, . . . , and $n_m$ **different** objects in the $m$th set.

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- If we need to select a pair of students (one from Aurora and another from Twinsburgh) in how many different ways we can do it? —> We may first select a student from Aurora and after (independently) select a student from Twinsburgh, thus, using Multiplication Principle, we get $3000 \times 4000 = 12000000$.

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# More Examples

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- What is the probability that there are all different values on the three dice (i.e. NO two dices give the same value)?

**Solution:** There are six outcomes of a single die.

Three dice are rolled one red, one blue and one green.

- How many different outcomes of this procedure are there?
- What is the probability that there are all different values on the three dice (i.e. NO two dices give the same value)?

**Solution:** There are six outcomes of a single die. So, by the multiplication principle the answer to the first question is $6 \times 6 \times 6 = 6^3 = 216$.

Three dice are rolled one red, one blue and one green.

- How many different outcomes of this procedure are there?
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We calculate the probability (answer to the second question) using the probability formula for an event - the fraction of outcomes producing the desired event divided by the all possible outcomes

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We calculate the probability (answer to the second question) using the probability formula for an event - the fraction of outcomes producing the desired event divided by the all possible outcomes (if you have never seen probability before NO WORRIES, treat this question simply asking what is the ratio of the number of all outcomes with different number on three dice to the number of all outcomes). The denominator of the fraction we are looking for is $6^3$. Now, to compute all possible ways to have no repetitions in our experiment we will imagine that the red die was rolled first (and thus there are 6 possible outcomes), next we roll the green die and thus there are 5 possible outcomes (remember green must be different from red!),

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$$\frac{6 \times 5 \times 4}{6^3} = \frac{5}{9}.$$

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$$30 \times 45 + 45 \times 12 + 30 \times 12 = 2250.$$

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$$75 + 15 + 1 = 91.$$

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$$75 + 15 + 1 = 91.$$

There is another nice way to answer this question.

How many ways are there to form a three-letter sequence using letters $a, b, c, d, e, f$?

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# More Examples

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The above solutions show that in many cases one may save a lot of time by finding a right way to model the problem. NOTE in many example it is very non-trivial to find "any" model and thus

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$$2^{70} - 1.$$