Lecture 13
MATH-42021/52021 Graph Theory and Combinatorics.

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$$\frac{11!}{4!\times 4!\times 2!\times 1!}.$$

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In how many different ways to select six hot-dogs are possible if there are three varieties of hot dog?

This is a selection-with-repetition problem, as always the main point is "to see it in a correct way".

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After end of Summer I semester and final exams in Artem's class, 5 students from Combinatorics, 5 from Differential Geometry and 5 from Probability class bought a block of 15 tickets for Kent Rock concert. If five seats are randomly selected for each class from the 15 seats in a row, what is the probability that Combinatorics students, Differential Geometry Students, and Probability students will each get a block of five consecutive seats?

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Solution: As we already discussed our goal it to find a fraction of "what we need" to all possible outcomes.

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$$3!/P(15;5,5,5).$$

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$$\binom{4}{2} P(10; 3, 3, 2, 2) + \binom{4}{1} P(10; 4, 2, 2, 2).$$

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$$\binom{10+3-1}{10} - \binom{5+3-1}{5}.$$