Lecture 13
MATH-42021/52021 Graph Theory and Combinatorics.

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Very VERY Famous Example

How many ways there are to arrange 11 letters in the word MISSISSIPPI?

Solution: There are 11 letters, but, yes, there is a problem: 4 of them are repeated "I", 4 of them repetitions of S and there are 2 repetitions of P so if we would answer $P(11, 11)$ we will over-calculate (much over-calculate).

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Theorem

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If there are n objects with r_1 of type 1, r_2 of type 2, ... and r_m of type m, where $r_1 + r_2 + r_3 + \cdots + r_m = n$, then the number of arrangements of these n objects, denoted by $P(n; r_1, r_2, r_3, \ldots, r_m)$ is

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This is a selection-with-repetition problem, as always the main point is "to see it in a correct way".

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Note that the above answer could be also seen as picking where 6 out of 8 places for x is (which is $\binom{8}{6}$). We can also note that in general the question was to select 6 objects out of 3 types and the answer is $\binom{6+3-1}{\ldots}$

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After end of Summer I semester and final exams in Artem's class, 5 students from Combinatorics, 5 from Differential Geometry and 5 from Probability class bought a block of 15 tickets for Kent Rock concert. If five seats are randomly selected for each class from the 15 seats in a row, what is the probability that Combinatorics students, Differential Geometry Students, and Probability students will each get a block of five consecutive seats?

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Solution: As we already discussed our goal it to find a fraction of "what we need" to all possible outcomes.

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$$3!/P(15; 5, 5, 5).$$

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$$\binom{4}{2} P(10; 3, 3, 2, 2) + \binom{4}{1} P(10; 4, 2, 2, 2).$$

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$$\binom{10+3-1}{10} - \binom{5+3-1}{5}.$$