Lecture 15
MATH-42021/52021 Graph Theory and Combinatorics.

Artem Zvavitch

Department of Mathematical Sciences, Kent State University

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An k-combination of n distinct objects is an unordered selection or subset of k out of the n objects. We will denote the number of such selection as $C(n, k)$:

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Binomial Idenitities

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$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \frac{n!}{(n-k)!(n-(n-k))!} = \binom{n}{n-k}.$$

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Pascal's triangle

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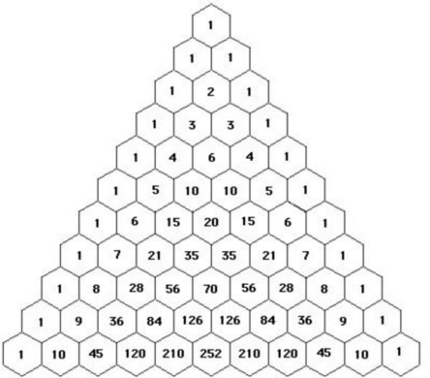

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Solution: Looks very different from previous problem?

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Now repeat with with $n=m+i$ and $k=i-1$ we get

$$\binom{m+i+1}{i}=\binom{m+i}{i}+\binom{m+i}{i-1}=\binom{m+i}{i}+\binom{m+i-1}{i-1}+\binom{m+i-1}{i-2}.$$

Just continue the process.

Prove that

$$\binom{m}{m}+\binom{m+1}{m}+\binom{m+2}{m}+\cdots+\binom{m+i}{m}=\binom{m+i+1}{i}.$$

Solution: Looks very different from previous problem? No, it is actually exactly the same!

Prove that
$$\binom{m}{0}+\binom{m+1}{1}+\binom{m+2}{2}+\cdots+\binom{m+i}{i}=\binom{m+i+1}{i}.$$

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Solution: Looks very different from previous problem? No, it is actually exactly the same! Just note that $\binom{m+i}{m}=\binom{m+i}{m+i-m}=\binom{m+i}{i}$.

Prove that

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Just continue the process.

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Solution: Looks very different from previous problem? No, it is actually exactly the same! Just note that $\binom{m+i}{m}=\binom{m+i}{m+i-m}=\binom{m+i}{i}$. In addition, we can also rewrite the above formula as

$$\binom{m}{m}+\binom{m+1}{m}+\binom{m+2}{m}+\cdots+\binom{n}{m}=\binom{n+1}{m+1},$$

where $n\geq m$.

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$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \cdots + \binom{n}{n}\binom{n}{n-n}$$

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This is already a cool formula! Can it get better?

Binomial Identities

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This is already a cool formula! Can it get better? YES! Use that $\binom{n}{k} = \binom{n}{n-k}$ to get that

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2.$$

Evaluate

$$1 \times 2 \times 3 + 2 \times 3 \times 4 + \cdots + (n-2) \times (n-1) \times n.$$

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$$2 \times 3 \times 4 = \frac{4!}{1!}$$

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OR

$$(k-2) \times (k-1) \times k = \frac{k!}{(k-3)!} = 3! \frac{k!}{3!(k-3)!} = 3! \binom{k}{3}.$$

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Evaluate
$$1\times 2\times 3+2\times 3\times 4+\cdots+(n-2)\times(n-1)\times n.$$

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$$1\times 2\times 3+2\times 3\times 4+\cdots+(n-2)\times(n-1)\times n=3!\left(\binom{3}{3}+\binom{4}{3}+\cdots+\binom{n}{3}\right).$$

Now search for possible identities, to get that

$$=3!\binom{n+1}{3+1}=6\binom{n+1}{4}.$$

Evaluate

$$1 + 2 + 3 + \cdots + (n-2) + (n-1) + n.$$

Binomial Identities.

Evaluate

$$1+2+3+\cdots+(n-2)+(n-1)+n.$$

Solution:

$$1+2+3+\cdots+(n-2)+(n-1)+n=\frac{1!}{1!}+\frac{2!}{1!}+\frac{3!}{2!}+\cdots+\frac{n!}{(n-1)!}$$

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Evaluate
$$1^2+2^2+3^2+\cdots+(n-1)^2+n^2.$$

Solution:

$$\sum_{k=1}^{n}k^2=\sum_{k=1}^{n}[k(k-1)+k]=\sum_{k=2}^{n}k(k-1)+\sum_{k=1}^{n}k.$$

Evaluate

$$1+2+3+\cdots+(n-2)+(n-1)+n.$$

Solution:

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We know the answer for the second sum (it is $(n+1)n/2$), so we concentrate on the first one:

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$$\sum_{k=2}^{n}k(k-1)=\sum_{k=2}^{n}\frac{k!}{(k-2)!}=2!\sum_{k=2}^{n}\frac{k!}{2!(k-2)!}=2!\sum_{k=2}^{n}\binom{k}{2}=2!\binom{n+1}{3}$$

Evaluate

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The final answer is

$$2!\frac{(n+1)!}{3!(n-2)!}+\frac{(n+1)n}{2}=\frac{(n+1)n(n-1)}{3}+\frac{(n+1)n}{2}=\frac{(n+1)n}{6}(2n-2+3)=\frac{(n+1)n(2n+1)}{6}.$$

Binomial Identities.

Evaluate

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$$1+2+3+\cdots+(n-2)+(n-1)+n=\frac{1!}{1!}+\frac{2!}{1!}+\frac{3!}{2!}+\cdots+\frac{n!}{(n-1)!}$$

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