Lecture 15
MATH-42021/52021 Graph Theory and Combinatorics.

Artem Zvavitch

Department of Mathematical Sciences, Kent State University

August, 2018.

An k-combination of n distinct objects is an unordered selection or subset of k out of the n objects. We will denote the number of such selection as $C(n, k)$:

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

An k-combination of n distinct objects is an unordered selection or subset of k out of the n objects. We will denote the number of such selection as $C(n,k)$:

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Let us compute $(x+y)^n$ we know that

An k-combination of n distinct objects is an unordered selection or subset of k out of the n objects. We will denote the number of such selection as $C(n,k)$:

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Let us compute $(x+y)^n$ we know that

$$(x+y)^n = \underbrace{(x+y)\times(x+y)\times(x+y)\times\cdots\times(x+y)}_{n-\text{times}}$$

An k-combination of n distinct objects is an unordered selection or subset of k out of the n objects. We will denote the number of such selection as $C(n,k)$:

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Let us compute $(x+y)^n$ we know that

$$(x+y)^n = \underbrace{(x+y)\times(x+y)\times(x+y)\times\cdots\times(x+y)}_{n-\text{times}}$$

Now we need to multiply, we need to pick ONE element from each of multipliers $(x+y)$ it can be x or y

An k-combination of n distinct objects is an unordered selection or subset of k out of the n objects. We will denote the number of such selection as $C(n, k)$:

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Let us compute $(x+y)^n$ we know that

$$(x+y)^n = \underbrace{(x+y) \times (x+y) \times (x+y) \times \cdots \times (x+y)}_{n-\text{times}}$$

Now we need to multiply, we need to pick ONE element from each of multipliers $(x+y)$ it can be x or y if we pick k times - x,

An k-combination of n distinct objects is an unordered selection or subset of k out of the n objects. We will denote the number of such selection as $C(n,k)$:

$$C(n,k)=\binom{n}{k}=\frac{n!}{k!(n-k)!}.$$

Let us compute $(x+y)^n$ we know that

$$(x+y)^n=\underbrace{(x+y)\times(x+y)\times(x+y)\times\cdots\times(x+y)}_{n-\text{times}}$$

Now we need to multiply, we need to pick ONE element from each of multipliers $(x+y)$ it can be x or y if we pick k times - x, then we must pick $(n-k)$ - y, and the outcome is $x^k y^{n-k}$.

An k-combination of n distinct objects is an unordered selection or subset of k out of the n objects. We will denote the number of such selection as $C(n,k)$:

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Let us compute $(x+y)^n$ we know that

$$(x+y)^n = \underbrace{(x+y)\times(x+y)\times(x+y)\times\cdots\times(x+y)}_{n-\text{times}}$$

Now we need to multiply, we need to pick ONE element from each of multipliers $(x+y)$ it can be x or y if we pick k times - x, then we must pick $(n-k)$ - y, and the outcome is $x^k y^{n-k}$. But in how many ways we can pick a place from which we pick k of x's?????

An k-combination of n distinct objects is an unordered selection or subset of k out of the n objects. We will denote the number of such selection as $C(n,k)$:

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Let us compute $(x+y)^n$ we know that

$$(x+y)^n = \underbrace{(x+y) \times (x+y) \times (x+y) \times \cdots \times (x+y)}_{n-\text{times}}$$

Now we need to multiply, we need to pick ONE element from each of multipliers $(x+y)$ it can be x or y if we pick k times - x, then we must pick $(n-k)$ - y, and the outcome is $x^k y^{n-k}$. But in how many ways we can pick a place from which we pick k of x's????? Note, we do note care about the order! Thus exactly from

$$\binom{n}{k}$$

places!!

An k-combination of n distinct objects is an unordered selection or subset of k out of the n objects. We will denote the number of such selection as $C(n,k)$:

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Let us compute $(x+y)^n$ we know that

$$(x+y)^n = \underbrace{(x+y)\times(x+y)\times(x+y)\times\cdots\times(x+y)}_{n-\text{times}}$$

Now we need to multiply, we need to pick ONE element from each of multipliers $(x+y)$ it can be x or y if we pick k times - x, then we must pick $(n-k)$ - y, and the outcome is $x^k y^{n-k}$. But in how many ways we can pick a place from which we pick k of x's????? Note, we do note care about the order! Thus exactly from

$$\binom{n}{k}$$

places!! And the final formula is

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Very simple but still cool:

$$\binom{n}{k} = \binom{n}{n-k}$$

Very simple but still cool:

$$\binom{n}{k} = \binom{n}{n-k}$$

Proof: Yes it is trivial , but still

Very simple but still cool:

$$\binom{n}{k} = \binom{n}{n-k}$$

Proof: Yes it is trivial , but still

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \frac{n!}{(n-k)!(n-(n-k))!} = \binom{n}{n-k}.$$

Also simple and even more cool:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Also simple and even more cool:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Proof:

$$\begin{aligned}
\binom{n-1}{k} + \binom{n-1}{k-1} &= \frac{(n-1)!}{k!(n-1-k)!} + \frac{(n-1)!}{(k-1)!(n-k)!} \\
&= \frac{(n-1)!}{(k-1)!(n-1-k)!}\left(\frac{1}{k} + \frac{1}{(n-k)}\right) \\
&= \frac{(n-1)!}{(k-1)!(n-1-k)!}\left(\frac{n-k+k}{k(n-k)}\right) \\
&= \frac{n!}{k!(n-k)!}
\end{aligned}$$

Also simple and even more cool:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Proof:

$$\begin{aligned}
\binom{n-1}{k} + \binom{n-1}{k-1} &= \frac{(n-1)!}{k!(n-1-k)!} + \frac{(n-1)!}{(k-1)!(n-k)!} \\
&= \frac{(n-1)!}{(k-1)!(n-1-k)!}\left(\frac{1}{k} + \frac{1}{(n-k)}\right) \\
&= \frac{(n-1)!}{(k-1)!(n-1-k)!}\left(\frac{n-k+k}{k(n-k)}\right) \\
&= \frac{n!}{k!(n-k)!}
\end{aligned}$$

We can also give a combinatorial proof.

Also simple and even more cool:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Proof:

$$\begin{aligned}\binom{n-1}{k} + \binom{n-1}{k-1} &= \frac{(n-1)!}{k!(n-1-k)!} + \frac{(n-1)!}{(k-1)!(n-k)!} \\ &= \frac{(n-1)!}{(k-1)!(n-1-k)!}\left(\frac{1}{k} + \frac{1}{(n-k)}\right) \\ &= \frac{(n-1)!}{(k-1)!(n-1-k)!}\left(\frac{n-k+k}{k(n-k)}\right) \\ &= \frac{n!}{k!(n-k)!}\end{aligned}$$

We can also give a combinatorial proof. Remember $C(n, k)$ is a number of ways to choose k (say) people out of n people to create a committee.

Also simple and even more cool:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Proof:

$$\begin{aligned}\binom{n-1}{k} + \binom{n-1}{k-1} &= \frac{(n-1)!}{k!(n-1-k)!} + \frac{(n-1)!}{(k-1)!(n-k)!} \\ &= \frac{(n-1)!}{(k-1)!(n-1-k)!}\left(\frac{1}{k} + \frac{1}{(n-k)}\right) \\ &= \frac{(n-1)!}{(k-1)!(n-1-k)!}\left(\frac{n-k+k}{k(n-k)}\right) \\ &= \frac{n!}{k!(n-k)!}\end{aligned}$$

We can also give a combinatorial proof. Remember $C(n,k)$ is a number of ways to choose k (say) people out of n people to create a committee. Now, assume, Artem is among those n people.

Also simple and even more cool:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Proof:

$$\begin{aligned}\binom{n-1}{k} + \binom{n-1}{k-1} &= \frac{(n-1)!}{k!(n-1-k)!} + \frac{(n-1)!}{(k-1)!(n-k)!} \\ &= \frac{(n-1)!}{(k-1)!(n-1-k)!}\left(\frac{1}{k} + \frac{1}{(n-k)}\right) \\ &= \frac{(n-1)!}{(k-1)!(n-1-k)!}\left(\frac{n-k+k}{k(n-k)}\right) \\ &= \frac{n!}{k!(n-k)!}\end{aligned}$$

We can also give a combinatorial proof. Remember $C(n,k)$ is a number of ways to choose k (say) people out of n people to create a committee. Now, assume, Artem is among those n people. Then there are two (disjoint!) ways to create a committee one with Artem in the committee another without him.

Also simple and even more cool:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Proof:

$$\begin{aligned}\binom{n-1}{k} + \binom{n-1}{k-1} &= \frac{(n-1)!}{k!(n-1-k)!} + \frac{(n-1)!}{(k-1)!(n-k)!} \\ &= \frac{(n-1)!}{(k-1)!(n-1-k)!}\left(\frac{1}{k} + \frac{1}{(n-k)}\right) \\ &= \frac{(n-1)!}{(k-1)!(n-1-k)!}\left(\frac{n-k+k}{k(n-k)}\right) \\ &= \frac{n!}{k!(n-k)!}\end{aligned}$$

We can also give a combinatorial proof. Remember $C(n,k)$ is a number of ways to choose k (say) people out of n people to create a committee. Now, assume, Artem is among those n people. Then there are two (disjoint!) ways to create a committee one with Artem in the committee another without him. Let's first compute how many different committees we can create without Artem's participation

Also simple and even more cool:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Proof:

$$\begin{aligned}\binom{n-1}{k} + \binom{n-1}{k-1} &= \frac{(n-1)!}{k!(n-1-k)!} + \frac{(n-1)!}{(k-1)!(n-k)!} \\ &= \frac{(n-1)!}{(k-1)!(n-1-k)!}\left(\frac{1}{k} + \frac{1}{(n-k)}\right) \\ &= \frac{(n-1)!}{(k-1)!(n-1-k)!}\left(\frac{n-k+k}{k(n-k)}\right) \\ &= \frac{n!}{k!(n-k)!}\end{aligned}$$

We can also give a combinatorial proof. Remember $C(n,k)$ is a number of ways to choose k (say) people out of n people to create a committee. Now, assume, Artem is among those n people. Then there are two (disjoint!) ways to create a committee one with Artem in the committee another without him. Let's first compute how many different committees we can create without Artem's participation – we need to choose k people out of $n-1$ people so there are $\binom{n-1}{k}$ ways to do it.

Also simple and even more cool:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Proof:

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \frac{(n-1)!}{k!(n-1-k)!} + \frac{(n-1)!}{(k-1)!(n-k)!}$$
$$= \frac{(n-1)!}{(k-1)!(n-1-k)!}\left(\frac{1}{k} + \frac{1}{(n-k)}\right)$$
$$= \frac{(n-1)!}{(k-1)!(n-1-k)!}\left(\frac{n-k+k}{k(n-k)}\right)$$
$$= \frac{n!}{k!(n-k)!}$$

We can also give a combinatorial proof. Remember $C(n,k)$ is a number of ways to choose k (say) people out of n people to create a committee. Now, assume, Artem is among those n people. Then there are two (disjoint!) ways to create a committee one with Artem in the committee another without him. Let's first compute how many different committees we can create without Artem's participation – we need to choose k people out of $n-1$ people so there are $\binom{n-1}{k}$ ways to do it. Next, in how many ways you can create a committee with Artem's participation:

Also simple and even more cool:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Proof:

$$\begin{aligned}\binom{n-1}{k} + \binom{n-1}{k-1} &= \frac{(n-1)!}{k!(n-1-k)!} + \frac{(n-1)!}{(k-1)!(n-k)!}\\ &= \frac{(n-1)!}{(k-1)!(n-1-k)!}\left(\frac{1}{k} + \frac{1}{(n-k)}\right)\\ &= \frac{(n-1)!}{(k-1)!(n-1-k)!}\left(\frac{n-k+k}{k(n-k)}\right)\\ &= \frac{n!}{k!(n-k)!}\end{aligned}$$

We can also give a combinatorial proof. Remember $C(n,k)$ is a number of ways to choose k (say) people out of n people to create a committee. Now, assume, Artem is among those n people. Then there are two (disjoint!) ways to create a committee one with Artem in the committee another without him. Let's first compute how many different committees we can create without Artem's participation – we need to choose k people out of $n-1$ people so there are $\binom{n-1}{k}$ ways to do it. Next, in how many ways you can create a committee with Artem's participation: Artem IS there so you left to choose $k-1$ people out of $n-1$ people: $\binom{n-1}{k-1}$ thus

Also simple and even more cool:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Proof:

$$\begin{aligned}\binom{n-1}{k} + \binom{n-1}{k-1} &= \frac{(n-1)!}{k!(n-1-k)!} + \frac{(n-1)!}{(k-1)!(n-k)!} \\ &= \frac{(n-1)!}{(k-1)!(n-1-k)!}\left(\frac{1}{k} + \frac{1}{(n-k)}\right) \\ &= \frac{(n-1)!}{(k-1)!(n-1-k)!}\left(\frac{n-k+k}{k(n-k)}\right) \\ &= \frac{n!}{k!(n-k)!}\end{aligned}$$

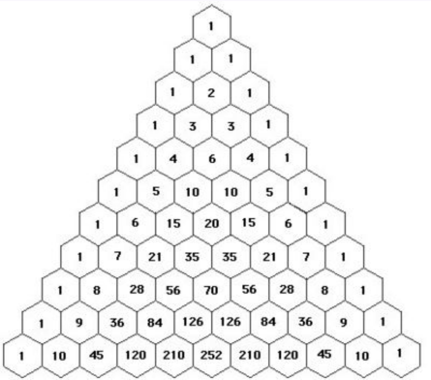
We can also give a combinatorial proof. Remember $C(n,k)$ is a number of ways to choose k (say) people out of n people to create a committee. Now, assume, Artem is among those n people. Then there are two (disjoint!) ways to create a committee one with Artem in the committee another without him. Let's first compute how many different committees we can create without Artem's participation – we need to choose k people out of $n-1$ people so there are $\binom{n-1}{k}$ ways to do it. Next, in how many ways you can create a committee with Artem's participation: Artem IS there so you left to choose $k-1$ people out of $n-1$ people: $\binom{n-1}{k-1}$ thus

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Pascal's triangle

Also simple and even more cool:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$



Binomial Identities: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

One can create a lot of cool identities just by using this formula.

Binomial Identities: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

One can create a lot of cool identities just by using this formula. Indeed, take $x = y = 1$:

$$2^n = \sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n}.$$

One can create a lot of cool identities just by using this formula. Indeed, take $x = y = 1$:

$$2^n = \sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n}.$$

Take $x = -1$, $y = 1$:

Binomial Identities: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

One can create a lot of cool identities just by using this formula. Indeed, take $x = y = 1$:

$$2^n = \sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n}.$$

Take $x = -1$, $y = 1$:

$$0 = \sum_{k=0}^{n} (-1)^k \binom{n}{k} = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^{n-1} \binom{n}{n-1} + (-1)^n \binom{n}{n}.$$

One can create a lot of cool identities just by using this formula. Indeed, take $x = y = 1$:

$$2^n = \sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n}.$$

Take $x = -1$, $y = 1$:

$$0 = \sum_{k=0}^{n} (-1)^k \binom{n}{k} = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^{n-1} \binom{n}{n-1} + (-1)^n \binom{n}{n}.$$

Now just take $y = 1$,

Binomial Identities: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

One can create a lot of cool identities just by using this formula. Indeed, take $x = y = 1$:

$$2^n = \sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n}.$$

Take $x = -1$, $y = 1$:

$$0 = \sum_{k=0}^{n} (-1)^k \binom{n}{k} = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^{n-1} \binom{n}{n-1} + (-1)^n \binom{n}{n}.$$

Now just take $y = 1$, you get $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$.

Binomial Identities: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

One can create a lot of cool identities just by using this formula. Indeed, take $x = y = 1$:

$$2^n = \sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n}.$$

Take $x = -1$, $y = 1$:

$$0 = \sum_{k=0}^{n} (-1)^k \binom{n}{k} = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^{n-1} \binom{n}{n-1} + (-1)^n \binom{n}{n}.$$

Now just take $y = 1$, you get $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$. Take the derivative from both sides of this equality:

$$n(1+x)^{n-1} = \sum_{k=1}^{n} \binom{n}{k} k x^{k-1}.$$

Now, substitute $x = 1$:

Binomial Identities: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

One can create a lot of cool identities just by using this formula. Indeed, take $x = y = 1$:

$$2^n = \sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n}.$$

Take $x = -1$, $y = 1$:

$$0 = \sum_{k=0}^{n} (-1)^k \binom{n}{k} = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^{n-1} \binom{n}{n-1} + (-1)^n \binom{n}{n}.$$

Now just take $y = 1$, you get $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$. Take the derivative from both sides of this equality:

$$n(1+x)^{n-1} = \sum_{k=1}^{n} \binom{n}{k} k x^{k-1}.$$

Now, substitute $x = 1$:

$$n2^{n-1} = \sum_{k=1}^{n} \binom{n}{k} k = 1\binom{n}{1} + 2\binom{n}{2} + \cdots + (n-1)\binom{n}{n-1} + n\binom{n}{n}.$$

Prove that

$$\binom{m}{0}+\binom{m+1}{1}+\binom{m+2}{2}+\cdots+\binom{m+i}{i}=\binom{m+i+1}{i}.$$

Prove that

$$\binom{m}{0}+\binom{m+1}{1}+\binom{m+2}{2}+\cdots+\binom{m+i}{i}=\binom{m+i+1}{i}.$$

Solution: Using identity $\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}$, with $n=m+i+1$ and $k=i$ we get

Prove that

$$\binom{m}{0}+\binom{m+1}{1}+\binom{m+2}{2}+\cdots+\binom{m+i}{i}=\binom{m+i+1}{i}.$$

Solution: Using identity $\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}$, with $n=m+i+1$ and $k=i$ we get

$$\binom{m+i+1}{i}=\binom{m+i}{i}+\binom{m+i}{i-1}.$$

Binomial Identities

Prove that
$$\binom{m}{0}+\binom{m+1}{1}+\binom{m+2}{2}+\cdots+\binom{m+i}{i}=\binom{m+i+1}{i}.$$

Solution: Using identity $\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}$, with $n=m+i+1$ and $k=i$ we get
$$\binom{m+i+1}{i}=\binom{m+i}{i}+\binom{m+i}{i-1}.$$

Now repeat with with $n=m+i$ and $k=i-1$ we get
$$\binom{m+i+1}{i}=\binom{m+i}{i}+\binom{m+i}{i-1}=\binom{m+i}{i}+\binom{m+i-1}{i-1}+\binom{m+i-1}{i-2}.$$

Prove that
$$\binom{m}{0}+\binom{m+1}{1}+\binom{m+2}{2}+\cdots+\binom{m+i}{i}=\binom{m+i+1}{i}.$$

Solution: Using identity $\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}$, with $n=m+i+1$ and $k=i$ we get
$$\binom{m+i+1}{i}=\binom{m+i}{i}+\binom{m+i}{i-1}.$$

Now repeat with with $n=m+i$ and $k=i-1$ we get
$$\binom{m+i+1}{i}=\binom{m+i}{i}+\binom{m+i}{i-1}=\binom{m+i}{i}+\binom{m+i-1}{i-1}+\binom{m+i-1}{i-2}.$$

Just continue the process.

Prove that

$$\binom{m}{0}+\binom{m+1}{1}+\binom{m+2}{2}+\cdots+\binom{m+i}{i}=\binom{m+i+1}{i}.$$

Solution: Using identity $\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}$, with $n=m+i+1$ and $k=i$ we get

$$\binom{m+i+1}{i}=\binom{m+i}{i}+\binom{m+i}{i-1}.$$

Now repeat with with $n=m+i$ and $k=i-1$ we get

$$\binom{m+i+1}{i}=\binom{m+i}{i}+\binom{m+i}{i-1}=\binom{m+i}{i}+\binom{m+i-1}{i-1}+\binom{m+i-1}{i-2}.$$

Just continue the process.

Prove that

$$\binom{m}{m}+\binom{m+1}{m}+\binom{m+2}{m}+\cdots+\binom{m+i}{m}=\binom{m+i+1}{i}.$$

Prove that

$$\binom{m}{0}+\binom{m+1}{1}+\binom{m+2}{2}+\cdots+\binom{m+i}{i}=\binom{m+i+1}{i}.$$

Solution: Using identity $\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}$, with $n=m+i+1$ and $k=i$ we get

$$\binom{m+i+1}{i}=\binom{m+i}{i}+\binom{m+i}{i-1}.$$

Now repeat with with $n=m+i$ and $k=i-1$ we get

$$\binom{m+i+1}{i}=\binom{m+i}{i}+\binom{m+i}{i-1}=\binom{m+i}{i}+\binom{m+i-1}{i-1}+\binom{m+i-1}{i-2}.$$

Just continue the process.

Prove that

$$\binom{m}{m}+\binom{m+1}{m}+\binom{m+2}{m}+\cdots+\binom{m+i}{m}=\binom{m+i+1}{i}.$$

Solution: Looks very different from previous problem?

Prove that

$$\binom{m}{0}+\binom{m+1}{1}+\binom{m+2}{2}+\cdots+\binom{m+i}{i}=\binom{m+i+1}{i}.$$

Solution: Using identity $\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}$, with $n=m+i+1$ and $k=i$ we get

$$\binom{m+i+1}{i}=\binom{m+i}{i}+\binom{m+i}{i-1}.$$

Now repeat with with $n=m+i$ and $k=i-1$ we get

$$\binom{m+i+1}{i}=\binom{m+i}{i}+\binom{m+i}{i-1}=\binom{m+i}{i}+\binom{m+i-1}{i-1}+\binom{m+i-1}{i-2}.$$

Just continue the process.

Prove that

$$\binom{m}{m}+\binom{m+1}{m}+\binom{m+2}{m}+\cdots+\binom{m+i}{m}=\binom{m+i+1}{i}.$$

Solution: Looks very different from previous problem? No, it is actually exactly the same!

Prove that

$$\binom{m}{0}+\binom{m+1}{1}+\binom{m+2}{2}+\cdots+\binom{m+i}{i}=\binom{m+i+1}{i}.$$

Solution: Using identity $\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}$, with $n=m+i+1$ and $k=i$ we get

$$\binom{m+i+1}{i}=\binom{m+i}{i}+\binom{m+i}{i-1}.$$

Now repeat with with $n=m+i$ and $k=i-1$ we get

$$\binom{m+i+1}{i}=\binom{m+i}{i}+\binom{m+i}{i-1}=\binom{m+i}{i}+\binom{m+i-1}{i-1}+\binom{m+i-1}{i-2}.$$

Just continue the process.

Prove that

$$\binom{m}{m}+\binom{m+1}{m}+\binom{m+2}{m}+\cdots+\binom{m+i}{m}=\binom{m+i+1}{i}.$$

Solution: Looks very different from previous problem? No, it is actually exactly the same! Just note that $\binom{m+i}{m}=\binom{m+i}{m+i-m}=\binom{m+i}{i}$.

Prove that

$$\binom{m}{0}+\binom{m+1}{1}+\binom{m+2}{2}+\cdots+\binom{m+i}{i}=\binom{m+i+1}{i}.$$

Solution: Using identity $\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}$, with $n=m+i+1$ and $k=i$ we get

$$\binom{m+i+1}{i}=\binom{m+i}{i}+\binom{m+i}{i-1}.$$

Now repeat with with $n=m+i$ and $k=i-1$ we get

$$\binom{m+i+1}{i}=\binom{m+i}{i}+\binom{m+i}{i-1}=\binom{m+i}{i}+\binom{m+i-1}{i-1}+\binom{m+i-1}{i-2}.$$

Just continue the process.

Prove that

$$\binom{m}{m}+\binom{m+1}{m}+\binom{m+2}{m}+\cdots+\binom{m+i}{m}=\binom{m+i+1}{i}.$$

Solution: Looks very different from previous problem? No, it is actually exactly the same! Just note that $\binom{m+i}{m}=\binom{m+i}{m+i-m}=\binom{m+i}{i}$. In addition, we can also rewrite the above formula as

$$\binom{m}{m}+\binom{m+1}{m}+\binom{m+2}{m}+\cdots+\binom{n}{m}=\binom{n+1}{m+1},$$

where $n\geq m$.

Assume we have $2n$ objects.

Assume we have $2n$ objects. You need to select n of them, in how many ways you can do it?

Assume we have $2n$ objects. You need to select n of them, in how many ways you can do it? Clearly, in $\binom{2n}{n}$.

Assume we have $2n$ objects. You need to select n of them, in how many ways you can do it? Clearly, in $\binom{2n}{n}$. Can we compute it in another way?

Assume we have $2n$ objects. You need to select n of them, in how many ways you can do it? Clearly, in $\binom{2n}{n}$. Can we compute it in another way? Actually YES!

Assume we have $2n$ objects. You need to select n of them, in how many ways you can do it? Clearly, in $\binom{2n}{n}$. Can we compute it in another way? Actually YES! Divide the group into two groups of n objects.

Assume we have $2n$ objects. You need to select n of them, in how many ways you can do it? Clearly, in $\binom{2n}{n}$. Can we compute it in another way? Actually YES! Divide the group into two groups of n objects. Now, each time you need to create group of n objects you pick k from first group and $n-k$ from the second group.

Assume we have $2n$ objects. You need to select n of them, in how many ways you can do it? Clearly, in $\binom{2n}{n}$. Can we compute it in another way? Actually YES! Divide the group into two groups of n objects. Now, each time you need to create group of n objects you pick k from first group and $n-k$ from the second group. This should be done for each $k = 0, 1, 2, 3, \dots, n$.

Assume we have $2n$ objects. You need to select n of them, in how many ways you can do it? Clearly, in $\binom{2n}{n}$. Can we compute it in another way? Actually YES! Divide the group into two groups of n objects. Now, each time you need to create group of n objects you pick k from first group and $n-k$ from the second group. This should be done for each $k = 0, 1, 2, 3, \dots, n$. For each k we have $\binom{n}{k}\binom{n}{n-k}$ options.

Assume we have $2n$ objects. You need to select n of them, in how many ways you can do it? Clearly, in $\binom{2n}{n}$. Can we compute it in another way? Actually YES! Divide the group into two groups of n objects. Now, each time you need to create group of n objects you pick k from first group and $n-k$ from the second group. This should be done for each $k = 0, 1, 2, 3, \dots, n$. For each k we have $\binom{n}{k}\binom{n}{n-k}$ options. Thus

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \dots + \binom{n}{n}\binom{n}{n-n}$$

Assume we have $2n$ objects. You need to select n of them, in how many ways you can do it? Clearly, in $\binom{2n}{n}$. Can we compute it in another way? Actually YES! Divide the group into two groups of n objects. Now, each time you need to create group of n objects you pick k from first group and $n-k$ from the second group. This should be done for each $k = 0, 1, 2, 3, \dots, n$. For each k we have $\binom{n}{k}\binom{n}{n-k}$ options. Thus

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \dots + \binom{n}{n}\binom{n}{n-n}$$

This is already a cool formula! Can it get better?

Assume we have $2n$ objects. You need to select n of them, in how many ways you can do it? Clearly, in $\binom{2n}{n}$. Can we compute it in another way? Actually YES! Divide the group into two groups of n objects. Now, each time you need to create group of n objects you pick k from first group and $n-k$ from the second group. This should be done for each $k = 0, 1, 2, 3, \dots, n$. For each k we have $\binom{n}{k}\binom{n}{n-k}$ options. Thus

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \cdots + \binom{n}{n}\binom{n}{n-n}$$

This is already a cool formula! Can it get better? YES! Use that $\binom{n}{k} = \binom{n}{n-k}$ to get that

Assume we have $2n$ objects. You need to select n of them, in how many ways you can do it? Clearly, in $\binom{2n}{n}$. Can we compute it in another way? Actually YES! Divide the group into two groups of n objects. Now, each time you need to create group of n objects you pick k from first group and $n-k$ from the second group. This should be done for each $k = 0, 1, 2, 3, \ldots, n$. For each k we have $\binom{n}{k}\binom{n}{n-k}$ options. Thus

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \cdots + \binom{n}{n}\binom{n}{n-n}$$

This is already a cool formula! Can it get better? YES! Use that $\binom{n}{k} = \binom{n}{n-k}$ to get that

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2.$$

Evaluate

$$1 \times 2 \times 3 + 2 \times 3 \times 4 + \cdots + (n-2) \times (n-1) \times n.$$

Evaluate
$$1 \times 2 \times 3 + 2 \times 3 \times 4 + \cdots + (n-2) \times (n-1) \times n.$$

Solution Looks not related to our story and binomial coefficients?

Evaluate
$$1 \times 2 \times 3 + 2 \times 3 \times 4 + \cdots + (n-2) \times (n-1) \times n.$$

Solution Looks not related to our story and binomial coefficients? Look more careful!

$$1 \times 2 \times 3 = 3!$$

Evaluate

$$1 \times 2 \times 3 + 2 \times 3 \times 4 + \cdots + (n-2) \times (n-1) \times n.$$

Solution Looks not related to our story and binomial coefficients? Look more careful!

$$1 \times 2 \times 3 = 3!$$

$$2 \times 3 \times 4 = \frac{4!}{1!}$$

Evaluate

$$1 \times 2 \times 3 + 2 \times 3 \times 4 + \cdots + (n-2) \times (n-1) \times n.$$

Solution Looks not related to our story and binomial coefficients? Look more careful!

$$1 \times 2 \times 3 = 3!$$

$$2 \times 3 \times 4 = \frac{4!}{1!}$$

$$(k-2) \times (k-1) \times k = \frac{k!}{(k-3)!}.$$

Evaluate
$$1 \times 2 \times 3 + 2 \times 3 \times 4 + \cdots + (n-2) \times (n-1) \times n.$$

Solution Looks not related to our story and binomial coefficients? Look more careful!

$$1 \times 2 \times 3 = 3!$$

$$2 \times 3 \times 4 = \frac{4!}{1!}$$

$$(k-2) \times (k-1) \times k = \frac{k!}{(k-3)!}.$$

OR

$$(k-2) \times (k-1) \times k = \frac{k!}{(k-3)!} = 3! \frac{k!}{3!(k-3)!} = 3! \binom{k}{3}.$$

Evaluate

$$1 \times 2 \times 3 + 2 \times 3 \times 4 + \cdots + (n-2) \times (n-1) \times n.$$

Solution Looks not related to our story and binomial coefficients? Look more careful!

$$1 \times 2 \times 3 = 3!$$

$$2 \times 3 \times 4 = \frac{4!}{1!}$$

$$(k-2) \times (k-1) \times k = \frac{k!}{(k-3)!}.$$

OR

$$(k-2) \times (k-1) \times k = \frac{k!}{(k-3)!} = 3! \frac{k!}{3!(k-3)!} = 3! \binom{k}{3}.$$

SO

$$1 \times 2 \times 3 + 2 \times 3 \times 4 + \cdots + (n-2) \times (n-1) \times n = 3! \left(\binom{3}{3} + \binom{4}{3} + \cdots + \binom{n}{3} \right).$$

Evaluate
$$1 \times 2 \times 3 + 2 \times 3 \times 4 + \cdots + (n-2) \times (n-1) \times n.$$

Solution Looks not related to our story and binomial coefficients? Look more careful!

$$1 \times 2 \times 3 = 3!$$

$$2 \times 3 \times 4 = \frac{4!}{1!}$$

$$(k-2) \times (k-1) \times k = \frac{k!}{(k-3)!}.$$

OR

$$(k-2) \times (k-1) \times k = \frac{k!}{(k-3)!} = 3! \frac{k!}{3!(k-3)!} = 3! \binom{k}{3}.$$

SO

$$1 \times 2 \times 3 + 2 \times 3 \times 4 + \cdots + (n-2) \times (n-1) \times n = 3! \left(\binom{3}{3} + \binom{4}{3} + \cdots + \binom{n}{3} \right).$$

Now search for possible identities, to get that

$$= 3! \binom{n+1}{3+1} = 6 \binom{n+1}{4}.$$

Binomial Identities.

Evaluate

$$1+2+3+\cdots+(n-2)+(n-1)+n.$$

Evaluate
$$1+2+3+\cdots+(n-2)+(n-1)+n.$$

Solution:
$$1+2+3+\cdots+(n-2)+(n-1)+n=\frac{1!}{1!}+\frac{2!}{1!}+\frac{3!}{2!}+\cdots+\frac{n!}{(n-1)!}$$

Evaluate

$$1+2+3+\cdots+(n-2)+(n-1)+n.$$

Solution:

$$1+2+3+\cdots+(n-2)+(n-1)+n=\frac{1!}{1!}+\frac{2!}{1!}+\frac{3!}{2!}+\cdots+\frac{n!}{(n-1)!}$$

$$=\sum_{k=1}^{n}\binom{k}{1}=\binom{n+1}{2}=\frac{(n+1)!}{2!(n-1)!}=\frac{(n+1)n}{2}.$$

Evaluate

$$1+2+3+\cdots+(n-2)+(n-1)+n.$$

Solution:

$$1+2+3+\cdots+(n-2)+(n-1)+n=\frac{1!}{1!}+\frac{2!}{1!}+\frac{3!}{2!}+\cdots+\frac{n!}{(n-1)!}$$

$$=\sum_{k=1}^{n}\binom{k}{1}=\binom{n+1}{2}=\frac{(n+1)!}{2!(n-1)!}=\frac{(n+1)n}{2}.$$

Evaluate

$$1^2+2^2+3^2+\cdots+(n-1)^2+n^2.$$

Evaluate

$$1+2+3+\cdots+(n-2)+(n-1)+n.$$

Solution:

$$1+2+3+\cdots+(n-2)+(n-1)+n = \frac{1!}{1!}+\frac{2!}{1!}+\frac{3!}{2!}+\cdots+\frac{n!}{(n-1)!}$$

$$=\sum_{k=1}^{n}\binom{k}{1}=\binom{n+1}{2}=\frac{(n+1)!}{2!(n-1)!}=\frac{(n+1)n}{2}.$$

Evaluate

$$1^2+2^2+3^2+\cdots+(n-1)^2+n^2.$$

Solution:

$$\sum_{k=1}^{n}k^2=\sum_{k=1}^{n}[k(k-1)+k]=\sum_{k=2}^{n}k(k-1)+\sum_{k=1}^{n}k.$$

Evaluate

$$1+2+3+\cdots+(n-2)+(n-1)+n.$$

Solution:

$$1+2+3+\cdots+(n-2)+(n-1)+n = \frac{1!}{1!}+\frac{2!}{1!}+\frac{3!}{2!}+\cdots+\frac{n!}{(n-1)!}$$

$$=\sum_{k=1}^{n}\binom{k}{1}=\binom{n+1}{2}=\frac{(n+1)!}{2!(n-1)!}=\frac{(n+1)n}{2}.$$

Evaluate

$$1^2+2^2+3^2+\cdots+(n-1)^2+n^2.$$

Solution:

$$\sum_{k=1}^{n}k^2=\sum_{k=1}^{n}[k(k-1)+k]=\sum_{k=2}^{n}k(k-1)+\sum_{k=1}^{n}k.$$

We know the answer for the second sum (it is $(n+1)n/2$), so we concentrate on the first one:

Binomial Identities.

Evaluate

$$1+2+3+\cdots+(n-2)+(n-1)+n.$$

Solution:

$$1+2+3+\cdots+(n-2)+(n-1)+n=\frac{1!}{1!}+\frac{2!}{1!}+\frac{3!}{2!}+\cdots+\frac{n!}{(n-1)!}$$

$$=\sum_{k=1}^{n}\binom{k}{1}=\binom{n+1}{2}=\frac{(n+1)!}{2!(n-1)!}=\frac{(n+1)n}{2}.$$

Evaluate

$$1^2+2^2+3^2+\cdots+(n-1)^2+n^2.$$

Solution:

$$\sum_{k=1}^{n}k^2=\sum_{k=1}^{n}[k(k-1)+k]=\sum_{k=2}^{n}k(k-1)+\sum_{k=1}^{n}k.$$

We know the answer for the second sum (it is $(n+1)n/2$), so we concentrate on the first one:

$$\sum_{k=2}^{n}k(k-1)=\sum_{k=2}^{n}\frac{k!}{(k-2)!}=2!\sum_{k=2}^{n}\frac{k!}{2!(k-2)!}=2!\sum_{k=2}^{n}\binom{k}{2}=2!\binom{n+1}{3}$$

Evaluate

$$1+2+3+\cdots+(n-2)+(n-1)+n.$$

Solution:

$$1+2+3+\cdots+(n-2)+(n-1)+n = \frac{1!}{1!}+\frac{2!}{1!}+\frac{3!}{2!}+\cdots+\frac{n!}{(n-1)!}$$

$$=\sum_{k=1}^{n}\binom{k}{1}=\binom{n+1}{2}=\frac{(n+1)!}{2!(n-1)!}=\frac{(n+1)n}{2}.$$

Evaluate

$$1^2+2^2+3^2+\cdots+(n-1)^2+n^2.$$

Solution:

$$\sum_{k=1}^{n}k^2=\sum_{k=1}^{n}[k(k-1)+k]=\sum_{k=2}^{n}k(k-1)+\sum_{k=1}^{n}k.$$

We know the answer for the second sum (it is $(n+1)n/2$), so we concentrate on the first one:

$$\sum_{k=2}^{n}k(k-1)=\sum_{k=2}^{n}\frac{k!}{(k-2)!}=2!\sum_{k=2}^{n}\frac{k!}{2!(k-2)!}=2!\sum_{k=2}^{n}\binom{k}{2}=2!\binom{n+1}{3}$$

The final answer is

$$2!\frac{(n+1)!}{3!(n-2)!}+\frac{(n+1)n}{2}=\frac{(n+1)n(n-1)}{3}+\frac{(n+1)n}{2}=\frac{(n+1)n}{6}(2n-2+3)=\frac{(n+1)n(2n+1)}{6}.$$

Binomial Identities.

Evaluate

$$1+2+3+\cdots+(n-2)+(n-1)+n.$$

Solution:

$$1+2+3+\cdots+(n-2)+(n-1)+n=\frac{1!}{1!}+\frac{2!}{1!}+\frac{3!}{2!}+\cdots+\frac{n!}{(n-1)!}$$

$$=\sum_{k=1}^{n}\binom{k}{1}=\binom{n+1}{2}=\frac{(n+1)!}{2!(n-1)!}=\frac{(n+1)n}{2}.$$

Evaluate

$$1^2+2^2+3^2+\cdots+(n-1)^2+n^2.$$

Solution:

$$\sum_{k=1}^{n}k^2=\sum_{k=1}^{n}[k(k-1)+k]=\sum_{k=2}^{n}k(k-1)+\sum_{k=1}^{n}k.$$

We know the answer for the second sum (it is $(n+1)n/2$), so we concentrate on the first one:

$$\sum_{k=2}^{n}k(k-1)=\sum_{k=2}^{n}\frac{k!}{(k-2)!}=2!\sum_{k=2}^{n}\frac{k!}{2!(k-2)!}=2!\sum_{k=2}^{n}\binom{k}{2}=2!\binom{n+1}{3}$$

The final answer is

$$2!\frac{(n+1)!}{3!(n-2)!}+\frac{(n+1)n}{2}=\frac{(n+1)n(n-1)}{3}+\frac{(n+1)n}{2}=\frac{(n+1)n}{6}(2n-2+3)=\frac{(n+1)n(2n+1)}{6}.$$