Lecture 16
MATH-42021/52021 Graph Theory and Combinatorics.

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Using Binomial theorem/formula we can write it in much more compact form $g(x) = (1+x)^n$.

Generating Functions

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Finally we get that the generating function we were looking for is $g(x) = \frac{1 - x^{n+1}}{1 - x}$.

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For example if $x = 1/2$ we get

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Operations with Generating Functions

Addition

Assume that $G(x)$ is the generating function for a sequence $a_0, a_1, a_2, \ldots$ and $F(x)$ is the generating function for a sequence$b_0, b_1, b_2, \ldots$. Then $G(x) + F(x)$ is the generating function for a sequence $a_0 + b_0, a_1 + b_1, a_2 + b_2, \ldots$

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$$G(x) = 7 + 7x + 7x^2 + 7x^3 + 7x^4 + \frac{x^5}{1-x}.$$

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Take the derivative from both sides, i.e.

$$\frac{d}{dx}\left(1 + x + x^2 + \cdots + x^{n-1} + x^n + \dots\right) = \frac{d}{dx}\left(\frac{1}{1-x}\right).$$

Derivative Generating Functions

We know that

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Now do standard computations:

$$1 + 2x + 3x^2 \cdots + (n-1)x^{n-2} + nx^{n-1} + \cdots = \frac{1}{(1-x)^2}.$$

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Take the derivative from both sides, i.e.

$$\frac{d}{dx}\left(1 + x + x^2 + \cdots + x^{n-1} + x^n + \dots\right) = \frac{d}{dx}\left(\frac{1}{1-x}\right).$$

Now do standard computations:

$$1 + 2x + 3x^2 \cdots + (n-1)x^{n-2} + nx^{n-1} + \cdots = \frac{1}{(1-x)^2}.$$

Assume that $G(x)$ is the generating function for a sequence $a_0, a_1, a_2, \dots$. Then $\frac{d}{dx}G(x)$ is a generating function of

$$a_1, 2a_2, \dots, ka_k, \dots$$

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$$G(x) = \frac{d}{dx}\left(x\frac{d}{dx}\left(\frac{1}{1-x}\right)\right) = \frac{d}{dx}\left(\frac{x}{(1-x)^2}\right) = \left(\frac{(1-x)^2 + 2x(1-x)}{(1-x)^4}\right)$$

$$= \frac{1+x}{(1-x)^3}.$$