Lecture 17
MATH-42021/52021 Graph Theory and Combinatorics.

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Consider a sequence of numbers $a_0, a_1, a_2, \ldots, a_n$. The generating function $g(x)$ is a polynomial defined as

$$g(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n.$$

Note that we also can consider an infinite sequence of numbers $a_0, a_1, a_2, \ldots, a_n, \ldots$. Then the generating function $g(x)$ is a series which is defined as

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Super Cool Application

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$$\frac{x}{(x + \frac{1+\sqrt{5}}{2})(x + \frac{1-\sqrt{5}}{2})} = \frac{A}{x + \frac{1+\sqrt{5}}{2}} + \frac{B}{x + \frac{1-\sqrt{5}}{2}}.$$

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Next you simply the right-hand side (just add up the fraction) compare the numerator to the numerator of the left-hand side and find A and B.

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SO

$$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{2}{\sqrt{5}-1}\right)^n - \left(\frac{-2}{\sqrt{5}+1}\right)^n\right]$$

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Example

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Thus a_n is simply the coefficient of $x^{l_1+l_2+l_3+l_4} = x^n$ in $(1 + x + x^2 + x^3)^4$ and the generating function we are looking for is

$$G(x) = (1 + x + x^2 + x^3)^4.$$

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Thus the generating function is

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