Lecture 1.1, MATH-42021/52021 Graph Theory and Combinatorics.

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"Introduction" to graphs.

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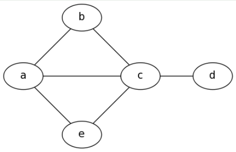

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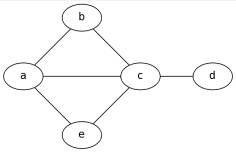

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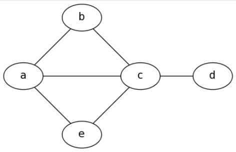

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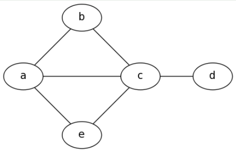

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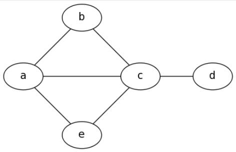

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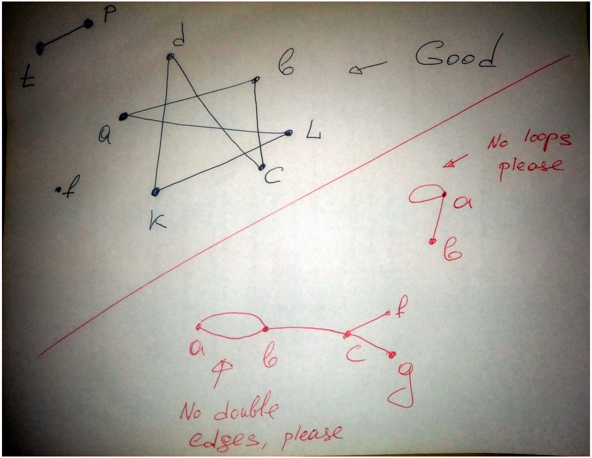
- If (a, b) is an edge. We call a, b - ends of the edge (a, b). We agree that ends of an edge must be different! (i.e. $a \neq b$).
- We also agree that there is at most one edge between any two vertices. (yes this comes from "simple")

More Examples.



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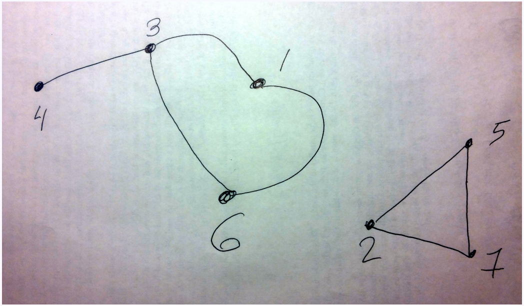
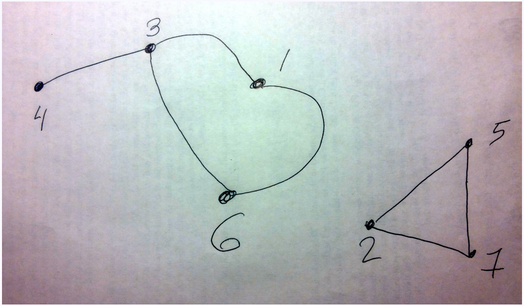
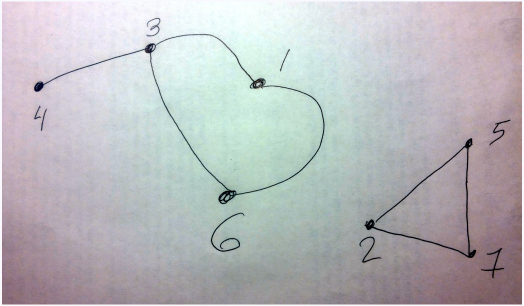




- a and b are adjacent vertices if they are connected by edge (i.e. if they are the end points of an edge, i.e. if $(a, b) \in E$). Example: 2 and 5 are adjacent (because $(2, 5) \in E$), but 4 and 1 are not, because $(4, 1) \notin E$.



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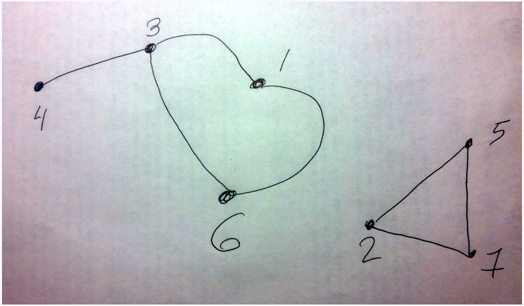

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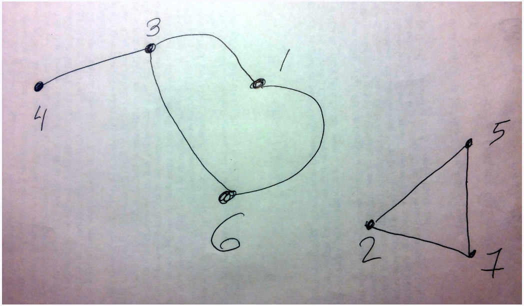
- **Path:** is a sequence of *distinct* vertices $(x_1, x_2, \ldots, x_n)$ such that consecutive vertices are adjacent. Example: $(4, 3, 1, 6)$ or $(2, 5, 7)$. Note: $(3, 6, 1, 4)$ not a path. $(4, 3, 1, 6, 3)$ not a path. And there is no pass from 2 to 1.



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- **Circuit**: is a pass that ends where it starts (i.e. $x_1 = x_n$, yes this is a bit of mix up because we repeated the last vertex, but this is allowed for this very special case and for only those vertices). Example: $(3,1,6,3)$, $(2,5,7,2)$.

Every Spring faculty of Math. Dep. of Aurora State University submit their request for available classes to teach over the summer, there are five people and there are five classes. The problem is to find one-to-one matching of Professors to classes or to show that such matching does not exist (and to ask Prof. to be more reasonable with their requests).

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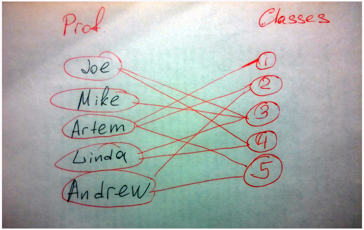

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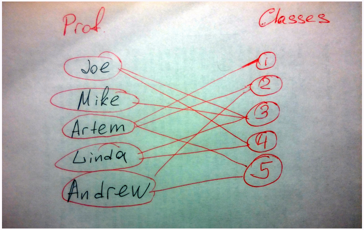

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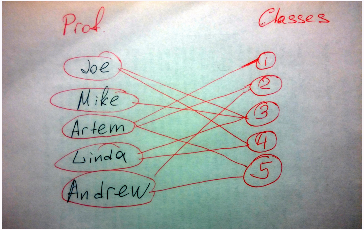

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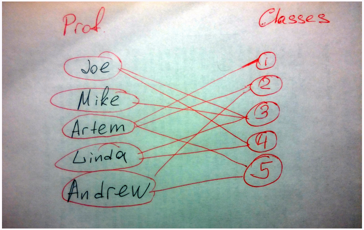
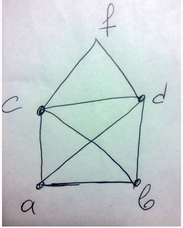


So does it exists a one-to-one matching of Professors to classes in the above graph? Unfortunately, NO. (just Look at what Joe, Linda and Mike want). The above graph is an example of a **bipartite** graph (is a graph whose vertices can be divided into two disjoint sets and such that every edge connects a vertex from one set to another - i.e. for each edge the end points belong to different sets). We will sure talk much more about those guys.

Suppose we are given a graph representing a network of telephone lines. We are interested in networks' vulnerability to accidental distraction we want to identify those lines and switching centers that must stay in service to avoid disconnecting the network.



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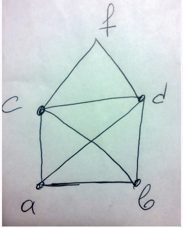

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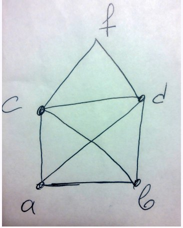

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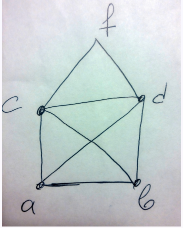

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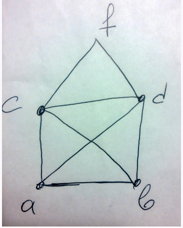

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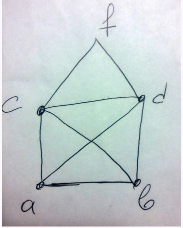

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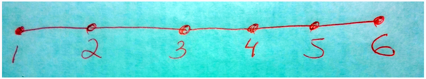

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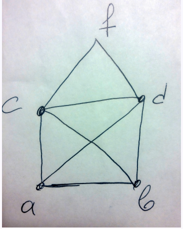

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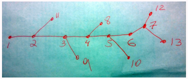

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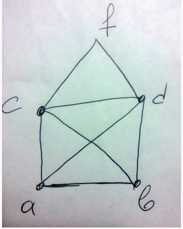

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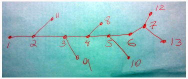

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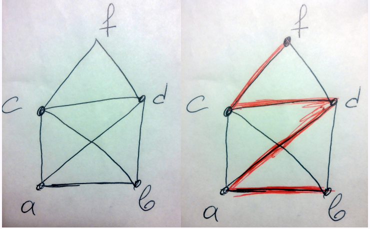

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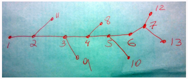

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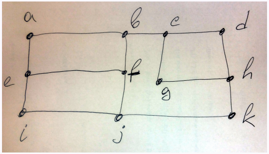
The graph below represents a part of a city's map. We want to position police at corners (vertices) so that they can keep every block (edge) under surveillance (or mathematically, every edge should have a police at at least one of its vertices). What is the fewest number of police that can do the job?



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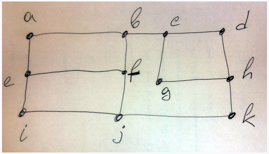

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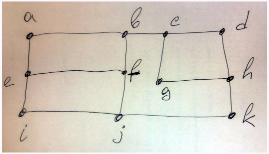

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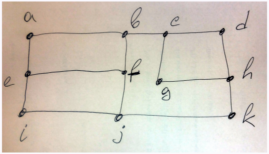

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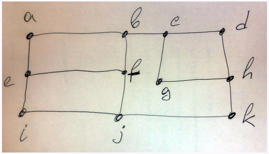

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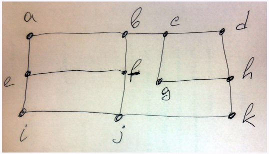

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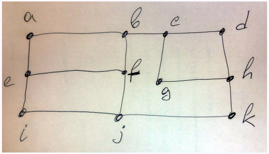

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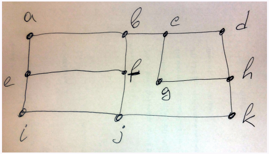

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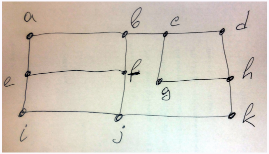

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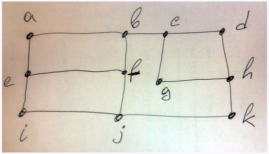

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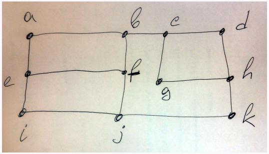

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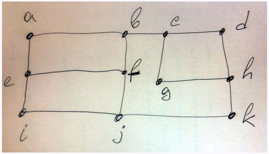

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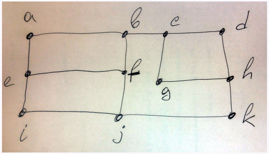

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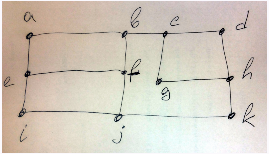

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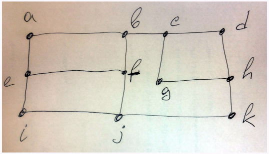

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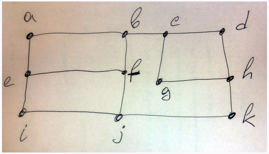

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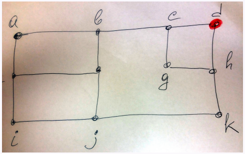

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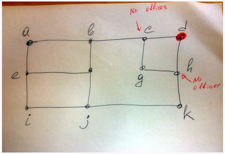

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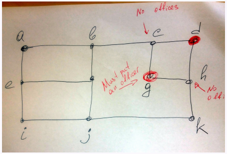

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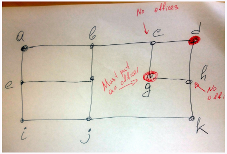

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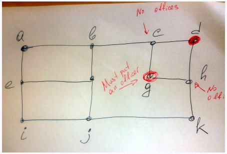

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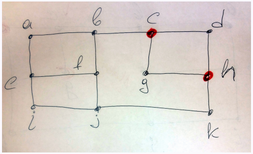

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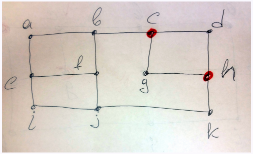

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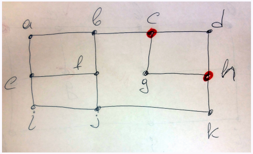

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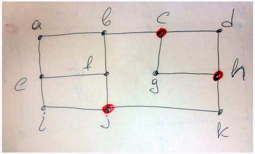

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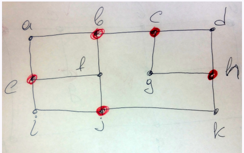
The graph below represents a part of a city's map. We want to position police at corners (vertices) so that they can keep every block (edge) under surveillance (or mathematically, every edge should have a police at at least one of its vertices). What is the fewest number of police that can do the job?



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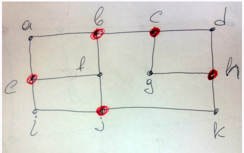

Police and edge cover (definition)

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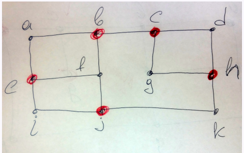

Edge Cover

A set C of vertices (i.e. $C \subset V$) in graph G with property that every edge of G is incident to at least one vertex in C is called an edge cover.

In the above graph we see the edge cover of minimal size.

Assume Math. Dep. at Aurora State University has a following schedule problem. As any department they have a lot of committees that meet for one hour each week. One wants to schedule of committee meeting times that minimizes the total number of hours but such that two committees with overlapping members do not meet at the same time.

Committee meetings and independent set of vertices

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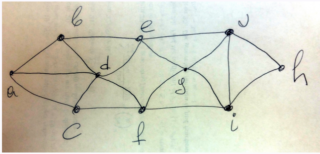

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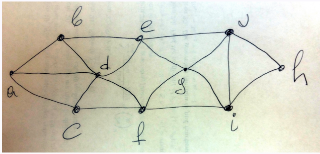

A set of committees can all meet at the same time IF there are no edges between the corresponding set of vertices.

Independent set

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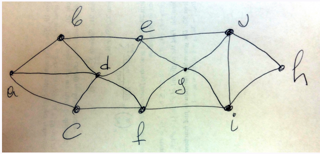

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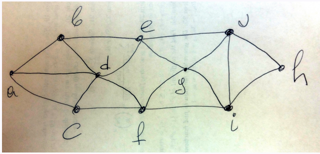

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We want to minimize the total number of hours! So we want as many as possible committees meet at the same time. Thus we need to find largest possible independent set(s). This is far from trivial. If we play a bit with the above example we will see that that there are two independent sets of size 4: a, e, f, h and b, c, g, h and every other independent set will have at most 3 vertices.

Edge Cover and Independent set

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Now assume $V \setminus I$ is an edge cover. Then every edge must have a vertex from $V \setminus I$ and so we can not have an edge with both ends from I, thus no two vertices from I are connected by an edge.

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A very interesting outcome of this theorem is that if I is an independent set of largest size in G then $V \setminus I$ is an edge cover of smallest possible size.

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A very interesting outcome of this theorem is that if I is an independent set of largest size in G then $V \setminus I$ is an edge cover of smallest possible size. SO finding maximal independent set is equivalent to finding a minimal edge cover.