Lecture 1.1, MATH-42021/52021 Graph Theory and Combinatorics.

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"Introduction" to graphs.

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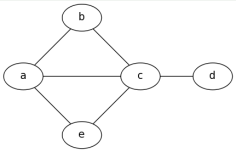

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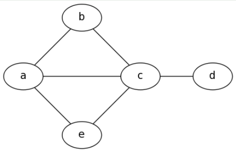

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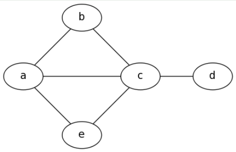

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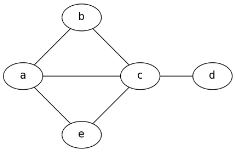

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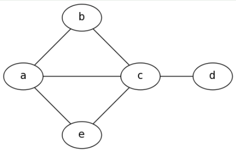

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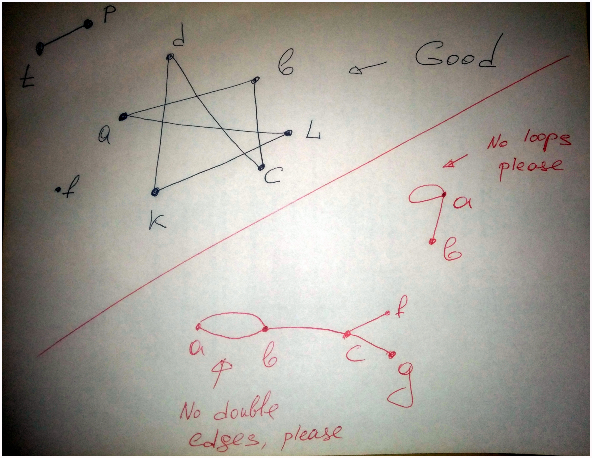
- We also agree that there is at most one edge between any two vertices. (yes this comes from "simple")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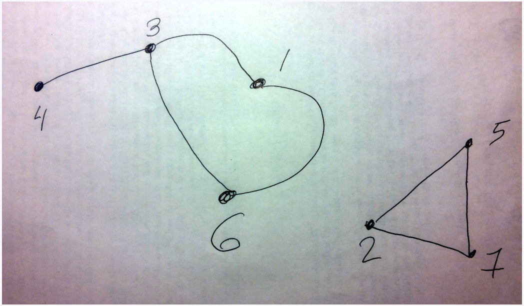
Bit more of definitions



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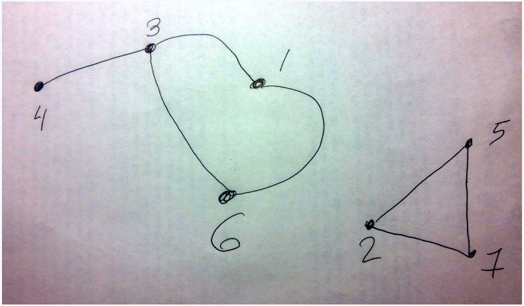

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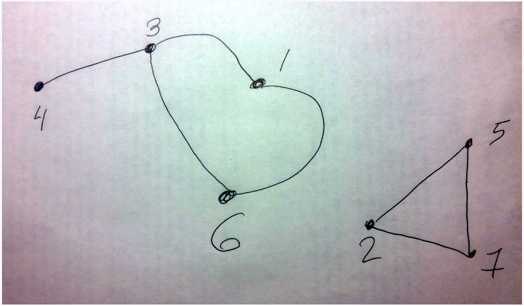

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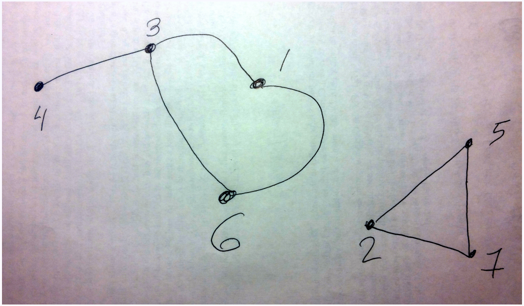

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- **Path:** is a sequence of *distinct* vertices $(x_1, x_2, \ldots, x_n)$ such that consecutive vertices are adjacent. Example: $(4, 3, 1, 6)$ or $(2, 5, 7)$. Note: $(3, 6, 1, 4)$ not a path. $(4, 3, 1, 6, 3)$ not a path. And there is no pass from 2 to 1.

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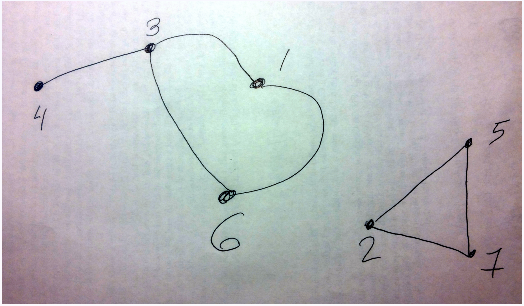

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- **Circuit**: is a pass that ends where it starts (i.e. $x_1 = x_n$, yes this is a bit of mix up because we repeated the last vertex, but this is allowed for this very special case and for only those vertices). Example: $(3,1,6,3)$, $(2,5,7,2)$.

Example: matching and bipartite

Every Spring faculty of Math. Dep. of Aurora State University submit their request for available classes to teach over the summer, there are five people and there are five classes. The problem is to find one-to-one matching of Professors to classes or to show that such matching does not exist (and to ask Prof. to be more reasonable with their requests).

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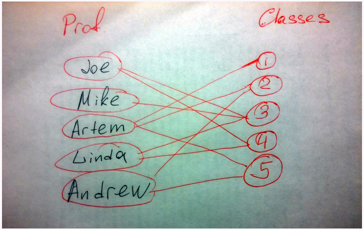

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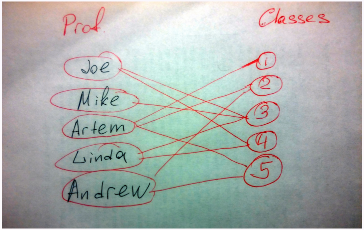

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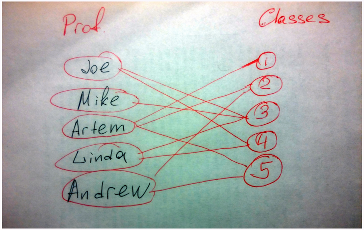

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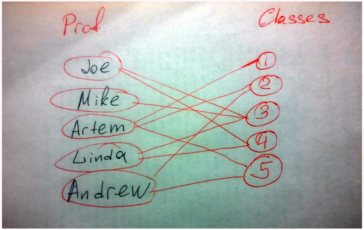

So does it exists a one-to-one matching of Professors to classes in the above graph? Unfortunately, NO. (just Look at what Joe, Linda and Mike want). The above graph is an example of a **bipartite** graph (is a graph whose vertices can be divided into two disjoint sets and such that every edge connects a vertex from one set to another - i.e. for each edge the end points belong to different sets). We will sure talk much more about those guys.

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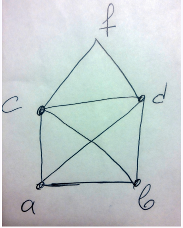
Suppose we are given a graph representing a network of telephone lines. We are interested in networks' vulnerability to accidental distraction we want to identify those lines and switching centers that must stay in service to avoid disconnecting the network.



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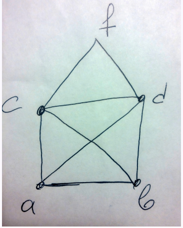

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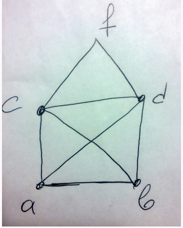

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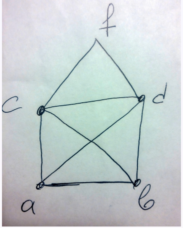

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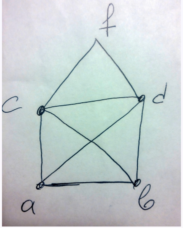

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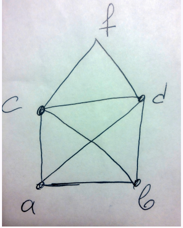

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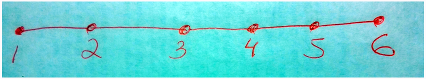

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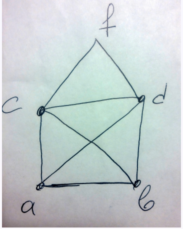

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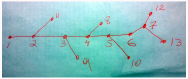

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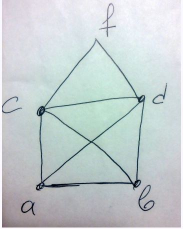

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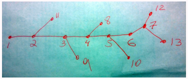

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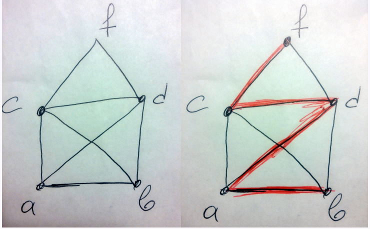

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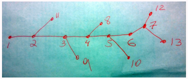

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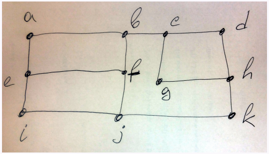
The graph below represents a part of a city's map. We want to position police at corners (vertices) so that they can keep every block (edge) under surveillance (or mathematically, every edge should have a police at at least one of its vertices). What is the fewest number of police that can do the job?



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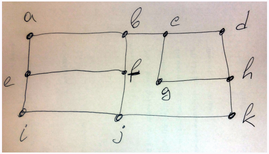

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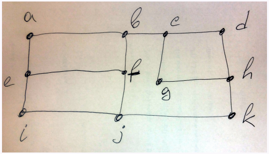

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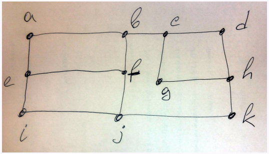

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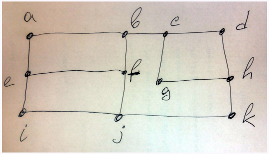

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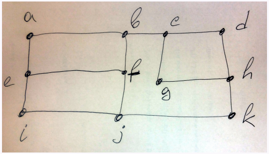

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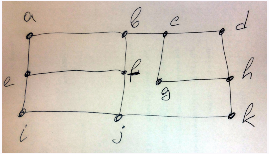

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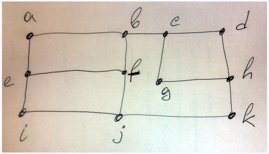

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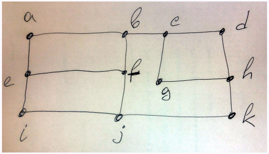

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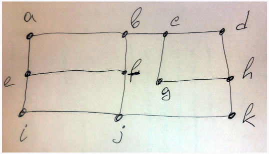

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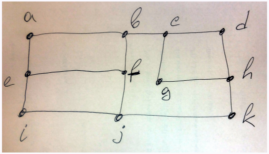

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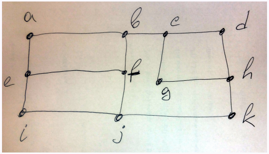

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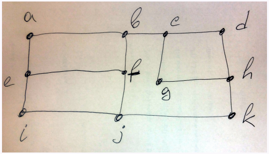

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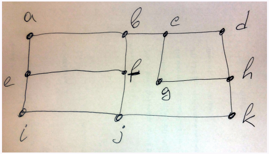

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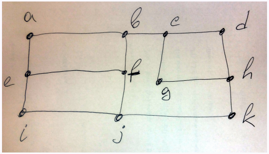

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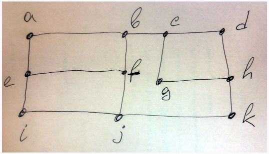

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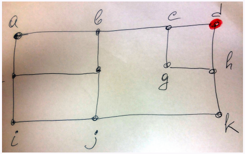

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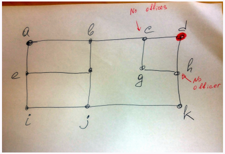

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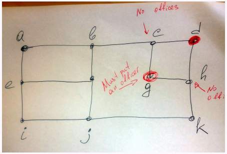

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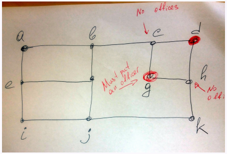

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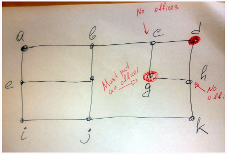

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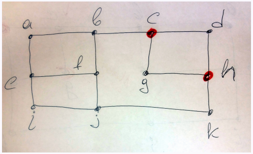

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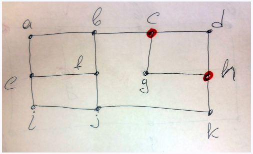

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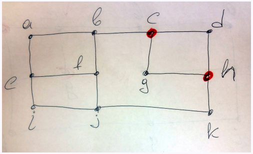

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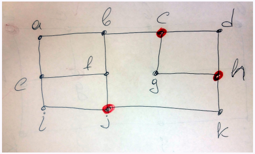

Lets first understand the minimal number of police we may need: The graph has 14 edges. Vertices b, c, e, f, h and h are the end of 3 edges each (i.e. each has degree 3). Vertices a, d, g, i and k are the end of 2 edges each (i.e. each has degree 2). Thus if put police at just 4 vertices they will NOT be able to do the job (indeed police can observe maximum 3 vertices from a given edge, thus 4 police can observe maximum $4 * 3 < 14$). So we need at least 5 police. IF we can find a good position of 5 police then we are done. If we put 5 police all at vertices of degree 3 then we cover $5 * 3 = 15$ edges and thus one edge must be covered twice (i.e. at both vertices). If 4 police at vertices of degree 3 and one of degree 2 then we cover at most 14 edges and must cover every edge exactly once. No other combination will do the job. Now we can start the systematic analysis (i.e. "very" clever guess). Start with an edge containing a vertex of degree 2 (say (c, d)). Assume there is an officer at vertex d. Then we can not put police at c or h (no edge can have both ends with police). Thus we MUST put police to vertex g. CONTRADICTION (we have two vertices of degree 2 with police). Thus we MUST start all over again and there is no police at d. But we MUST take care of edges (c, d) and (d, h). Now look at k we should not put police there! (because, if we do we have and edge with two police and a vertex of degree 2 with a police). Thus we must put police at j. We finish using the same reasoning.

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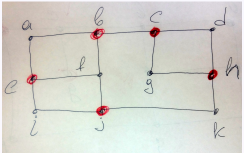

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Police and edge cover (definition)

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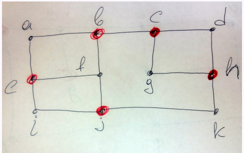

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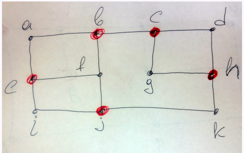

Edge Cover

A set C of vertices (i.e. $C \subset V$) in graph G with property that every edge of G is incident to at least one vertex in C is called an edge cover.

In the above graph we see the edge cover of minimal size.

Assume Math. Dep. at Aurora State University has a following schedule problem. As any department they have a lot of committees that meet for one hour each week. One wants to schedule of committee meeting times that minimizes the total number of hours but such that two committees with overlapping members do not meet at the same time.

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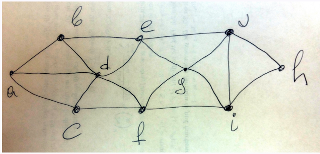

Committee meetings and independent set of vertices

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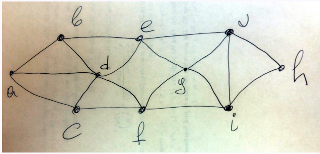

A set of committees can all meet at the same time IF there are no edges between the corresponding set of vertices.

Independent set

A set of vertices without an edge between any two is called an independent set of vertices.

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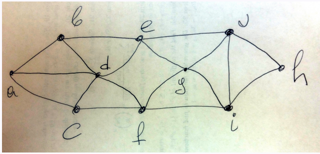

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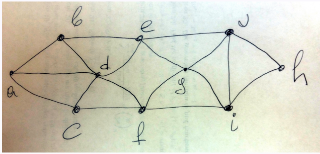

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We want to minimize the total number of hours! So we want as many as possible committees meet at the same time. Thus we need to find largest possible independent set(s). This is far from trivial. If we play a bit with the above example we will see that that there are two independent sets of size 4: a, e, f, h and b, c, g, h and every other independent set will have at most 3 vertices.

Theorem

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$\square$

A very interesting outcome of this theorem is that if I is an independent set of largest size in G then $V \setminus I$ is an edge cover of smallest possible size.

Edge Cover and Independent set

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