Lecture 3
MATH-42021/52021 Graph Theory and Combinatorics.

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Edge Counting

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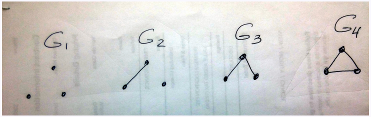

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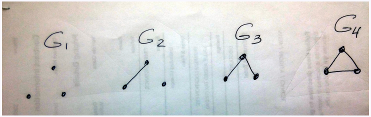

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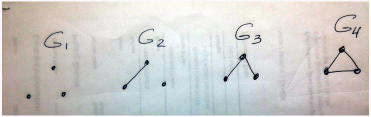

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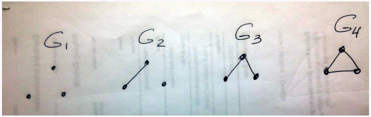

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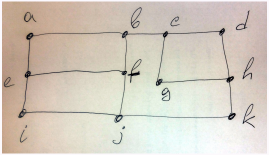

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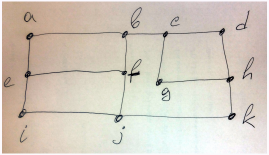

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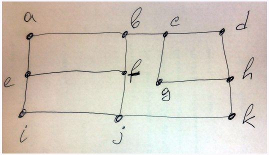

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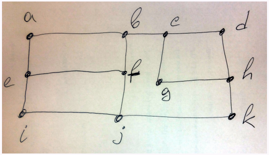

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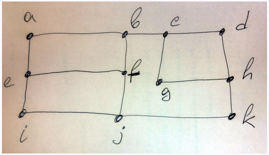

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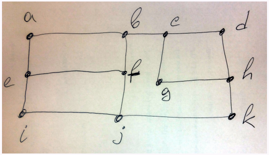

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Corollary:

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Edge Counting: Special Example

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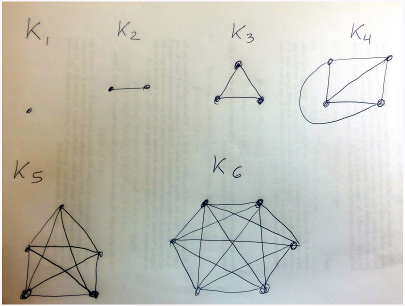

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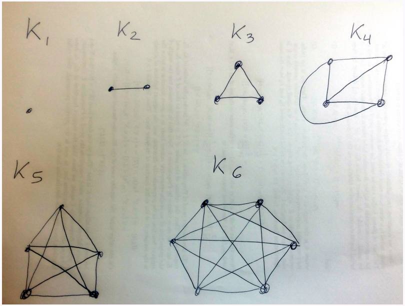

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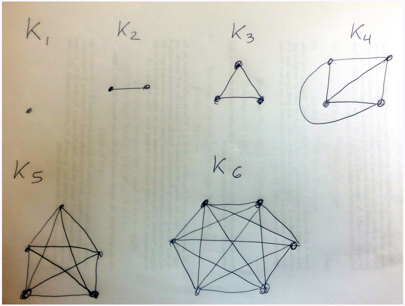

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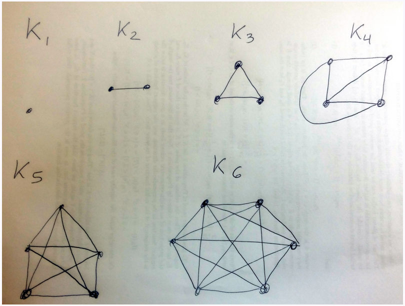

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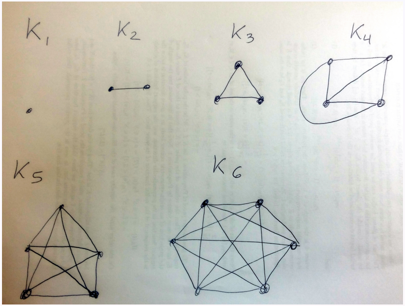

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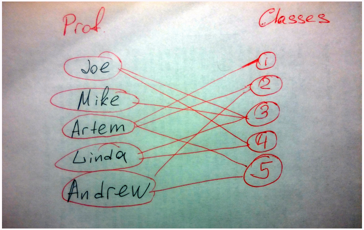
$$\frac{n(n-1)}{2}.$$

Bipartite Graph

A **bipartite graph** is a graph $G = (V, E)$ whose vertices V can be divided into two disjoint sets L, R (i.e. $V = L \cup R$ and $L \cap R = \emptyset$) such that L and R are independent sets (i.e. vertices in the each set L and R are not connected to each other) such that every edge connects a vertex from L to a vertex from R.



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Now assume that every circuit in G is of even length, our goal is to show that G is bipartite (It will be not a problem for us if there are no circuits (i.e. all of them of length zero), actually in this case the task is trivial). We may assume that G is connected (if not we deal with each part separately).

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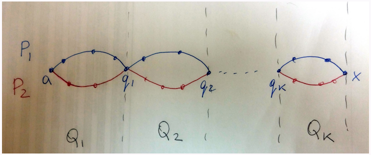
If P_1 and P_2 have common vertices other then a and x say $q_1, \dots q_k$ then



Theorem:

A graph G is bipartite is and only if any circuit in G has even length.

Let's start our construction/algorithm. Pick up any vertex, say a, and put it in L. Now we must put all vertices adjacent to a in R. Put all vertices which are two edges away from a to L, so in general if a vertex is odd steps (odd number of edges) from a put it into R if even number of edges from a put it into L.

One problem with this algorithm is why it is well defined? i.e. why there is no vertex x such that there is pass P_1 of even length from a to x and pass P_2 of odd length from a to x. Assume (towards a contradiction) that we actually have such a situation, then consider $P_1 \cup P_2$ it is a circuit and we can "play" with it.

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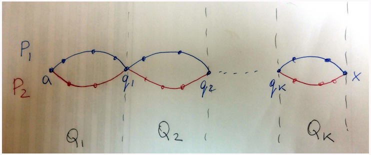

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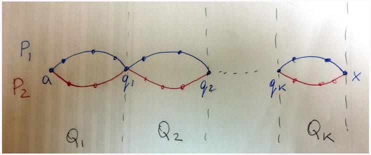

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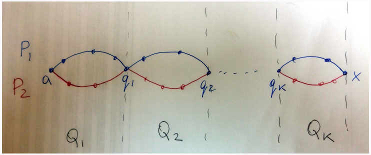

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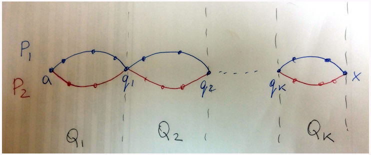

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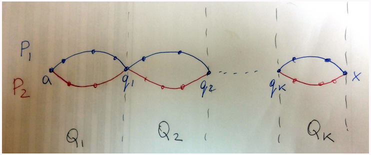

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Thus $\#P_1 + \#P_2 - 2 = \sum_{i=1}^{k} \#Q_i$ and $\sum_{i=1}^{k} \#Q_i$ is odd and THUS we must have at least one $\#Q_i$ be an odd number. So we found an odd length circuit which contradicts the assumption of

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$\square$

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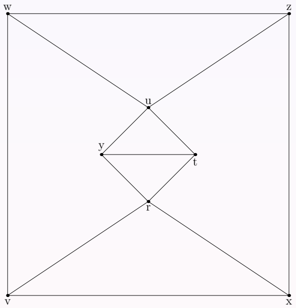
Consider a graph below, is it Bipartite?



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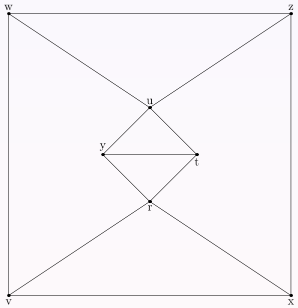

The answer is NO. We notice that there is an odd length circuit (y, u, t) and apply the theorem!