Lecture 3
MATH-42021/52021 Graph Theory and Combinatorics.

Artem Zvavitch

Department of Mathematical Sciences, Kent State University

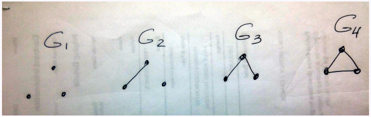
July, 2018.

Edge Counting

Is there is any connection between number of edges in a graph and a number of vertices?

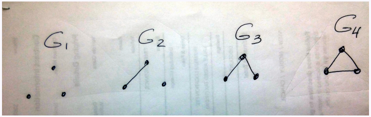
Edge Counting

Is there is any connection between number of edges in a graph and a number of vertices?



Looks like "no", but actually there are many nice formulas!!!

Is there is any connection between number of edges in a graph and a number of vertices?



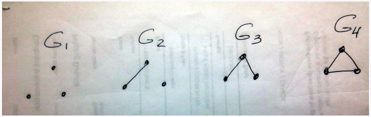
Looks like "no", but actually there are many nice formulas!!!

Theorem:

In any graph, the sum of the degrees of all vertices is equal to twice the number of edges.

Edge Counting

Is there is any connection between number of edges in a graph and a number of vertices?



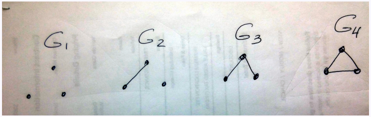
Looks like "no", but actually there are many nice formulas!!!

Theorem:

In any graph, the sum of the degrees of all vertices is equal to twice the number of edges.

Proof : When we sum up the degrees of all vertices we count all instances of some edges being incident at some vertex.

Is there is any connection between number of edges in a graph and a number of vertices?



Looks like "no", but actually there are many nice formulas!!!

Theorem:

In any graph, the sum of the degrees of all vertices is equal to twice the number of edges.

Proof : When we sum up the degrees of all vertices we count all instances of some edges being incident at some vertex. But an edge has TWO ends, so it is incident to TWO vertices, so we counted every edge twice.

□

Edge Counting

Theorem:

In any graph, the sum of the degrees of all vertices is equal to twice the number of edges.

Proof : When we sum up the degrees of all vertices we count all instances of some edges being incident at some vertex. But and edge has TWO ends, so it is incident to TWO vertices, so we counted every edge twice.

□

Edge Counting

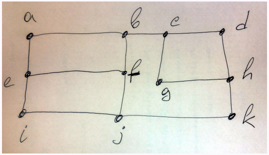
Theorem:

In any graph, the sum of the degrees of all vertices is equal to twice the number of edges.

Proof : When we sum up the degrees of all vertices we count all instances of some edges being incident at some vertex. But and edge has TWO ends, so it is incident to TWO vertices, so we counted every edge twice.

□

Let's check on another example



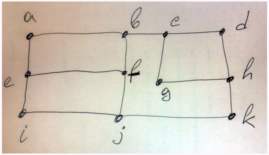
Theorem:

In any graph, the sum of the degrees of all vertices is equal to twice the number of edges.

Proof : When we sum up the degrees of all vertices we count all instances of some edges being incident at some vertex. But and edge has TWO ends, so it is incident to TWO vertices, so we counted every edge twice.

□

Let's check on another example



$deg(a) + deg(b) + deg(c) + deg(d) + deg(e) + deg(f) + deg(g) + deg(h) + deg(i) + deg(j) + deg(k)$

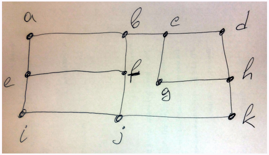
Theorem:

In any graph, the sum of the degrees of all vertices is equal to twice the number of edges.

Proof : When we sum up the degrees of all vertices we count all instances of some edges being incident at some vertex. But and edge has TWO ends, so it is incident to TWO vertices, so we counted every edge twice.

□

Let's check on another example



$$deg(a)+deg(b)+deg(c)+deg(d)+deg(e)+deg(f)+deg(g)+deg(h)+deg(i)+deg(j)+deg(k)$$

$$=2+3+3+2+3+3+2+3+2+3+2=28=2*\#E$$

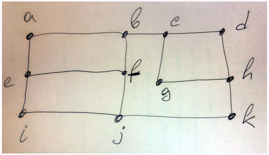
Theorem:

In any graph, the sum of the degrees of all vertices is equal to twice the number of edges.

Proof : When we sum up the degrees of all vertices we count all instances of some edges being incident at some vertex. But and edge has TWO ends, so it is incident to TWO vertices, so we counted every edge twice.

□

Let's check on another example



$$deg(a) + deg(b) + deg(c) + deg(d) + deg(e) + deg(f) + deg(g) + deg(h) + deg(i) + deg(j) + deg(k)$$

$$= 2 + 3 + 3 + 2 + 3 + 3 + 2 + 3 + 2 + 3 + 2 = 28 = 2 * \#E$$

Note, that by the theorem, the sum of degrees is always an even number.

Edge Counting

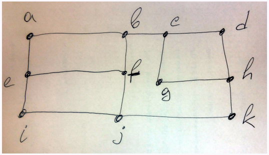
Theorem:

In any graph, the sum of the degrees of all vertices is equal to twice the number of edges.

Proof : When we sum up the degrees of all vertices we count all instances of some edges being incident at some vertex. But and edge has TWO ends, so it is incident to TWO vertices, so we counted every edge twice.

□

Let's check on another example



$$deg(a)+deg(b)+deg(c)+deg(d)+deg(e)+deg(f)+deg(g)+deg(h)+deg(i)+deg(j)+deg(k)$$

$$=2+3+3+2+3+3+2+3+2+3+2=28=2*\#E$$

Note, that by the theorem, the sum of degrees is always an even number. We may have some vertices with odd degrees, but to make the final sum even we must have an even number of them:

Edge Counting

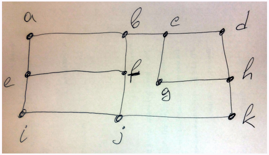
Theorem:

In any graph, the sum of the degrees of all vertices is equal to twice the number of edges.

Proof : When we sum up the degrees of all vertices we count all instances of some edges being incident at some vertex. But and edge has TWO ends, so it is incident to TWO vertices, so we counted every edge twice.

□

Let's check on another example



$deg(a) + deg(b) + deg(c) + deg(d) + deg(e) + deg(f) + deg(g) + deg(h) + deg(i) + deg(j) + deg(k)$

$$= 2 + 3 + 3 + 2 + 3 + 3 + 2 + 3 + 2 + 3 + 2 = 28 = 2 * \#E$$

Note, that by the theorem, the sum of degrees is always an even number. We may have some vertices with odd degrees, but to make the final sum even we must have an even number of them:

Corollary:

In any graph, the number of vertices of odd degree is even.

Edge Counting: Examples

Suppose we would like to construct a graph of 30 edges and have all vertices of degree 5. How many vertices must the graph have?

Suppose we would like to construct a graph of 30 edges and have all vertices of degree 5. How many vertices must the graph have?

Let v be a number of vertices,

Suppose we would like to construct a graph of 30 edges and have all vertices of degree 5. How many vertices must the graph have?

Let v be a number of vertices, then the sum of all degrees is $5v$,

Suppose we would like to construct a graph of 30 edges and have all vertices of degree 5. How many vertices must the graph have?

Let v be a number of vertices, then the sum of all degrees is $5v$, but by the theorem, it must be equal to twice the number of edges,

Suppose we would like to construct a graph of 30 edges and have all vertices of degree 5. How many vertices must the graph have?

Let v be a number of vertices, then the sum of all degrees is $5v$, but by the theorem, it must be equal to twice the number of edges, i.e. $5v = 60$ and $v = 12$.

Suppose we would like to construct a graph of 30 edges and have all vertices of degree 5. How many vertices must the graph have?

Let v be a number of vertices, then the sum of all degrees is $5v$, but by the theorem, it must be equal to twice the number of edges, i.e. $5v = 60$ and $v = 12$.

Is it possible to have a group of 7 people such that each person knows exactly 3 other people in the group?

Edge Counting: Examples

Suppose we would like to construct a graph of 30 edges and have all vertices of degree 5. How many vertices must the graph have?

Let v be a number of vertices, then the sum of all degrees is $5v$, but by the theorem, it must be equal to twice the number of edges, i.e. $5v = 60$ and $v = 12$.

Is it possible to have a group of 7 people such that each person knows exactly 3 other people in the group?

We use graphs to model this problem. Indeed, let vertices represent people and we put an edge between two vertices/people if they know each other.

Suppose we would like to construct a graph of 30 edges and have all vertices of degree 5. How many vertices must the graph have?

Let v be a number of vertices, then the sum of all degrees is $5v$, but by the theorem, it must be equal to twice the number of edges, i.e. $5v = 60$ and $v = 12$.

Is it possible to have a group of 7 people such that each person knows exactly 3 other people in the group?

We use graphs to model this problem. Indeed, let vertices represent people and we put an edge between two vertices/people if they know each other. So now the question is it possible to have a graph of 7 vertices, each vertex having degree 3?

Suppose we would like to construct a graph of 30 edges and have all vertices of degree 5. How many vertices must the graph have?

Let v be a number of vertices, then the sum of all degrees is $5v$, but by the theorem, it must be equal to twice the number of edges, i.e. $5v = 60$ and $v = 12$.

Is it possible to have a group of 7 people such that each person knows exactly 3 other people in the group?

We use graphs to model this problem. Indeed, let vertices represent people and we put an edge between two vertices/people if they know each other. So now the question is it possible to have a graph of 7 vertices, each vertex having degree 3? The answer is NO and it follows from the Corollary −> the number of vertices with odd degrees must be even.

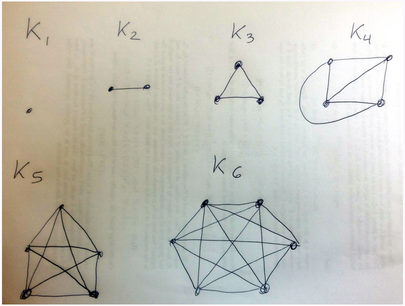
Edge Counting: Special Example

K_n - complete graph of n vertices

A graph of n vertices in which each vertex is adjacent to all the other vertices is called a complete graph of n vertices and denoted K_n.

K_n - complete graph of n vertices

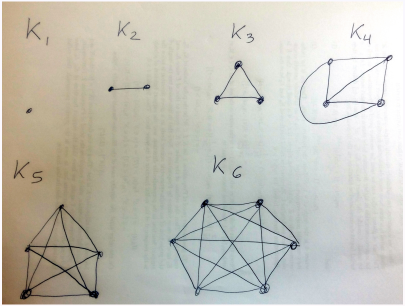
A graph of n vertices in which each vertex is adjacent to all the other vertices is called a complete graph of n vertices and denoted K_n.



Edge Counting: Special Example

K_n - complete graph of n vertices

A graph of n vertices in which each vertex is adjacent to all the other vertices is called a complete graph of n vertices and denoted K_n.

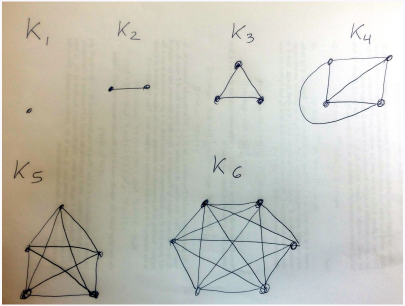


An interesting (combinatorial) question is how many edges K_n may have?

Edge Counting: Special Example

K_n - complete graph of n vertices

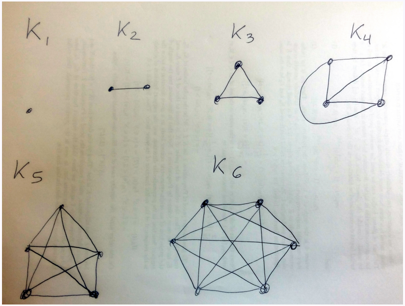
A graph of n vertices in which each vertex is adjacent to all the other vertices is called a complete graph of n vertices and denoted K_n.



An interesting (combinatorial) question is how many edges K_n may have? It has n vertices, each vertex has degree $n-1$ so sum of all degrees is $n(n-1)$,

K_n - complete graph of n vertices

A graph of n vertices in which each vertex is adjacent to all the other vertices is called a complete graph of n vertices and denoted K_n.

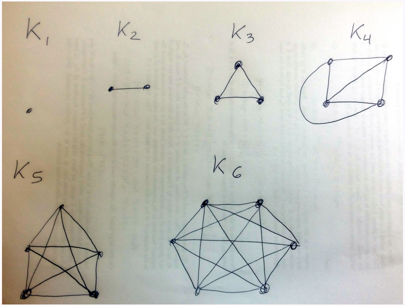


An interesting (combinatorial) question is how many edges K_n may have? It has n vertices, each vertex has degree $n-1$ so sum of all degrees is $n(n-1)$, which is, by the theorem, is twice the number of edges.

Edge Counting: Special Example

K_n - complete graph of n vertices

A graph of n vertices in which each vertex is adjacent to all the other vertices is called a complete graph of n vertices and denoted K_n.

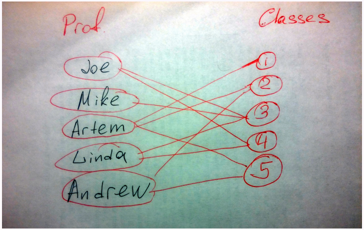


An interesting (combinatorial) question is how many edges K_n may have? It has n vertices, each vertex has degree $n-1$ so sum of all degrees is $n(n-1)$, which is, by the theorem, is twice the number of edges. Thus the number of edges in K_n is

$$\frac{n(n-1)}{2}.$$

Bipartite Graph

A **bipartite graph** is a graph $G = (V, E)$ whose vertices V can be divided into two disjoint sets L, R (i.e. $V = L \cup R$ and $L \cap R = \emptyset$) such that L and R are independent sets (i.e. vertices in the each set L and R are not connected to each other) such that every edge connects a vertex from L to a vertex from R.



Bipartite Graph: how to recognize it?

A **bipartite graph** is a graph $G = (V, E)$ whose vertices V can be divided into two disjoint sets L, R (i.e. $V = L \cup R$ and $L \cap R = \emptyset$) such that L and R are independent sets (i.e. vertices in the each set L and R are not connected to each other) such that every edge connects a vertex from L to a vertex from R.

Bipartite Graph: how to recognize it?

A **bipartite graph** is a graph $G = (V, E)$ whose vertices V can be divided into two disjoint sets L, R (i.e. $V = L \cup R$ and $L \cap R = \emptyset$) such that L and R are independent sets (i.e. vertices in the each set L and R are not connected to each other) such that every edge connects a vertex from L to a vertex from R.

The **length** of a circuit or path is the number of edges in it.

Bipartite Graph: how to recognize it?

A **bipartite graph** is a graph $G = (V, E)$ whose vertices V can be divided into two disjoint sets L, R (i.e. $V = L \cup R$ and $L \cap R = \emptyset$) such that L and R are independent sets (i.e. vertices in the each set L and R are not connected to each other) such that every edge connects a vertex from L to a vertex from R.

The **length** of a circuit or path is the number of edges in it.

Theorem:

A graph G is bipartite is and only if any circuit in G has even length.

Bipartite Graph: Proof of the characterization theorem.

Theorem:

A graph G is bipartite is and only if any circuit in G has even length.

Proof : Assume G is bipartite.

Theorem:

A graph G is bipartite is and only if any circuit in G has even length.

Proof : Assume G is bipartite. We need to show that all circuits are of even length.

Theorem:

A graph G is bipartite is and only if any circuit in G has even length.

Proof : Assume G is bipartite. We need to show that all circuits are of even length. If there are no circuits −> done (zero length is an even length).

Theorem:

A graph G is bipartite is and only if any circuit in G has even length.

Proof : Assume G is bipartite. We need to show that all circuits are of even length. If there are no circuits −> done (zero length is an even length). Assume there is a circuit $(x_1, x_2, \dots, x_{n-1}, x_n)$, where all x_i are different and x_n is adjacent to x_1.

Theorem:

A graph G is bipartite is and only if any circuit in G has even length.

Proof : Assume G is bipartite. We need to show that all circuits are of even length. If there are no circuits −> done (zero length is an even length). Assume there is a circuit $(x_1, x_2, \dots, x_{n-1}, x_n)$, where all x_i are different and x_n is adjacent to x_1. Note that x_1 must belong to L or R, without loss of generality we may assume that it belongs to L.

Theorem:

A graph G is bipartite is and only if any circuit in G has even length.

Proof : Assume G is bipartite. We need to show that all circuits are of even length. If there are no circuits –> done (zero length is an even length). Assume there is a circuit $(x_1, x_2, \ldots, x_{n-1}, x_n)$, where all x_i are different and x_n is adjacent to x_1. Note that x_1 must belong to L or R, without loss of generality we may assume that it belongs to L. But then x_2 must be in R,

Theorem:

A graph G is bipartite is and only if any circuit in G has even length.

Proof : Assume G is bipartite. We need to show that all circuits are of even length. If there are no circuits −> done (zero length is an even length). Assume there is a circuit $(x_1, x_2, \dots, x_{n-1}, x_n)$, where all x_i are different and x_n is adjacent to x_1. Note that x_1 must belong to L or R, without loss of generality we may assume that it belongs to L. But then x_2 must be in R, x_3 must be in L and so on

Theorem:

A graph G is bipartite is and only if any circuit in G has even length.

Proof : Assume G is bipartite. We need to show that all circuits are of even length. If there are no circuits −> done (zero length is an even length). Assume there is a circuit $(x_1, x_2, \dots, x_{n-1}, x_n)$, where all x_i are different and x_n is adjacent to x_1. Note that x_1 must belong to L or R, without loss of generality we may assume that it belongs to L. But then x_2 must be in R, x_3 must be in L and so on (so every $x_{2k+1} \in L$ and every $x_{2k} \in R$).

Theorem:

A graph G is bipartite is and only if any circuit in G has even length.

Proof : Assume G is bipartite. We need to show that all circuits are of even length. If there are no circuits −> done (zero length is an even length). Assume there is a circuit $(x_1, x_2, \ldots, x_{n-1}, x_n)$, where all x_i are different and x_n is adjacent to x_1. Note that x_1 must belong to L or R, without loss of generality we may assume that it belongs to L. But then x_2 must be in R, x_3 must be in L and so on (so every $x_{2k+1} \in L$ and every $x_{2k} \in R$). But, again, we are playing with the circuit, so x_n is adjacent to x_1, which is in L, thus, x_n must be in R and so n is even!

Theorem:

A graph G is bipartite is and only if any circuit in G has even length.

Proof : Assume G is bipartite. We need to show that all circuits are of even length. If there are no circuits −> done (zero length is an even length). Assume there is a circuit $(x_1, x_2, \dots, x_{n-1}, x_n)$, where all x_i are different and x_n is adjacent to x_1. Note that x_1 must belong to L or R, without loss of generality we may assume that it belongs to L. But then x_2 must be in R, x_3 must be in L and so on (so every $x_{2k+1} \in L$ and every $x_{2k} \in R$). But, again, we are playing with the circuit, so x_n is adjacent to x_1, which is in L, thus, x_n must be in R and so n is even!

Now assume that every circuit in G is of even length, our goal is to show that G is bipartite

Theorem:

A graph G is bipartite is and only if any circuit in G has even length.

Proof : Assume G is bipartite. We need to show that all circuits are of even length. If there are no circuits −> done (zero length is an even length). Assume there is a circuit $(x_1, x_2, \dots, x_{n-1}, x_n)$, where all x_i are different and x_n is adjacent to x_1. Note that x_1 must belong to L or R, without loss of generality we may assume that it belongs to L. But then x_2 must be in R, x_3 must be in L and so on (so every $x_{2k+1} \in L$ and every $x_{2k} \in R$). But, again, we are playing with the circuit, so x_n is adjacent to x_1, which is in L, thus, x_n must be in R and so n is even!

Now assume that every circuit in G is of even length, our goal is to show that G is bipartite (It will be not a problem for us if there are no circuits (i.e. all of them of length zero), actually in this case the task is trivial). We may assume that G is connected (if not we deal with each part separately).

Theorem:

A graph G is bipartite is and only if any circuit in G has even length.

Let's start our construction/algorithm.

Theorem:

A graph G is bipartite is and only if any circuit in G has even length.

Let's start our construction/algorithm. Pick up any vertex, say a, and put it in L.

Theorem:

A graph G is bipartite is and only if any circuit in G has even length.

Let's start our construction/algorithm. Pick up any vertex, say a, and put it in L. Now we must put all vertices adjacent to a in R.

Theorem:

A graph G is bipartite is and only if any circuit in G has even length.

Let's start our construction/algorithm. Pick up any vertex, say a, and put it in L. Now we must put all vertices adjacent to a in R. Put all vertices which are two edges away from a to L,

Theorem:

A graph G is bipartite is and only if any circuit in G has even length.

Let's start our construction/algorithm. Pick up any vertex, say a, and put it in L. Now we must put all vertices adjacent to a in R. Put all vertices which are two edges away from a to L, so in general if a vertex is odd steps (odd number of edges) from a put it into R if even number of edges from a put it into L.

Theorem:

A graph G is bipartite is and only if any circuit in G has even length.

Let's start our construction/algorithm. Pick up any vertex, say a, and put it in L. Now we must put all vertices adjacent to a in R. Put all vertices which are two edges away from a to L, so in general if a vertex is odd steps (odd number of edges) from a put it into R if even number of edges from a put it into L.
One problem with this algorithm is why it is well defined?

Theorem:

A graph G is bipartite is and only if any circuit in G has even length.

Let's start our construction/algorithm. Pick up any vertex, say a, and put it in L. Now we must put all vertices adjacent to a in R. Put all vertices which are two edges away from a to L, so in general if a vertex is odd steps (odd number of edges) from a put it into R if even number of edges from a put it into L.
One problem with this algorithm is why it is well defined? i.e. why there is no vertex x such that there is pass P_1 of even length from a to x and pass P_2 of odd length from a to x.

Theorem:

A graph G is bipartite is and only if any circuit in G has even length.

Let's start our construction/algorithm. Pick up any vertex, say a, and put it in L. Now we must put all vertices adjacent to a in R. Put all vertices which are two edges away from a to L, so in general if a vertex is odd steps (odd number of edges) from a put it into R if even number of edges from a put it into L.

One problem with this algorithm is why it is well defined? i.e. why there is no vertex x such that there is pass P_1 of even length from a to x and pass P_2 of odd length from a to x. Assume (towards a contradiction) that we actually have such a situation, then consider $P_1 \cup P_2$ it is a circuit and we can "play" with it.

Theorem:

A graph G is bipartite is and only if any circuit in G has even length.

Let's start our construction/algorithm. Pick up any vertex, say a, and put it in L. Now we must put all vertices adjacent to a in R. Put all vertices which are two edges away from a to L, so in general if a vertex is odd steps (odd number of edges) from a put it into R if even number of edges from a put it into L.
One problem with this algorithm is why it is well defined? i.e. why there is no vertex x such that there is pass P_1 of even length from a to x and pass P_2 of odd length from a to x. Assume (towards a contradiction) that we actually have such a situation, then consider $P_1 \cup P_2$ it is a circuit and we can "play" with it.
First, consider a case when P_1 and P_2 have no common vertices but a and x.

Theorem:

A graph G is bipartite is and only if any circuit in G has even length.

Let's start our construction/algorithm. Pick up any vertex, say a, and put it in L. Now we must put all vertices adjacent to a in R. Put all vertices which are two edges away from a to L, so in general if a vertex is odd steps (odd number of edges) from a put it into R if even number of edges from a put it into L.

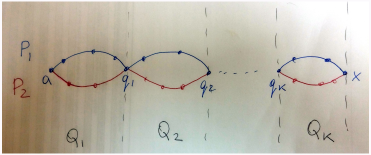
One problem with this algorithm is why it is well defined? i.e. why there is no vertex x such that there is pass P_1 of even length from a to x and pass P_2 of odd length from a to x. Assume (towards a contradiction) that we actually have such a situation, then consider $P_1 \cup P_2$ it is a circuit and we can "play" with it.

First, consider a case when P_1 and P_2 have no common vertices but a and x. Then there was an even number of vertices in P_1 and odd in P_2 so together it is ODD (we counted a and x twice which does not change even/odd) and we get a contradiction.

Theorem:

A graph G is bipartite is and only if any circuit in G has even length.

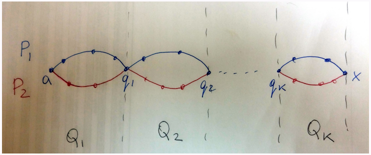
Let's start our construction/algorithm. Pick up any vertex, say a, and put it in L. Now we must put all vertices adjacent to a in R. Put all vertices which are two edges away from a to L, so in general if a vertex is odd steps (odd number of edges) from a put it into R if even number of edges from a put it into L.
One problem with this algorithm is why it is well defined? i.e. why there is no vertex x such that there is pass P_1 of even length from a to x and pass P_2 of odd length from a to x. Assume (towards a contradiction) that we actually have such a situation, then consider $P_1 \cup P_2$ it is a circuit and we can "play" with it.
First, consider a case when P_1 and P_2 have no common vertices but a and x. Then there was an even number of vertices in P_1 and odd in P_2 so together it is ODD (we counted a and x twice which does not change even/odd) and we get a contradiction.
If P_1 and P_2 have common vertices other then a and x say $q_1, \dots q_k$ then



Theorem:

A graph G is bipartite is and only if any circuit in G has even length.

Let's start our construction/algorithm. Pick up any vertex, say a, and put it in L. Now we must put all vertices adjacent to a in R. Put all vertices which are two edges away from a to L, so in general if a vertex is odd steps (odd number of edges) from a put it into R if even number of edges from a put it into L.
One problem with this algorithm is why it is well defined? i.e. why there is no vertex x such that there is pass P_1 of even length from a to x and pass P_2 of odd length from a to x. Assume (towards a contradiction) that we actually have such a situation, then consider $P_1 \cup P_2$ it is a circuit and we can "play" with it.
First, consider a case when P_1 and P_2 have no common vertices but a and x. Then there was an even number of vertices in P_1 and odd in P_2 so together it is ODD (we counted a and x twice which does not change even/odd) and we get a contradiction.
If P_1 and P_2 have common vertices other then a and x say $q_1, \dots q_k$ then



Notice that we can compute number of vertices in $P_1 \cup P_2$ in two ways:

$$\#P_1 + \#P_2 - 2 - k = \sum_{i=1}^{k} \#Q_i - k.$$

Theorem:

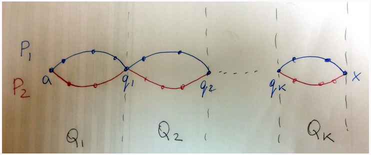
A graph G is bipartite is and only if any circuit in G has even length.

Let's start our construction/algorithm. Pick up any vertex, say a, and put it in L. Now we must put all vertices adjacent to a in R. Put all vertices which are two edges away from a to L, so in general if a vertex is odd steps (odd number of edges) from a put it into R if even number of edges from a put it into L.

One problem with this algorithm is why it is well defined? i.e. why there is no vertex x such that there is pass P_1 of even length from a to x and pass P_2 of odd length from a to x. Assume (towards a contradiction) that we actually have such a situation, then consider $P_1 \cup P_2$ it is a circuit and we can "play" with it.

First, consider a case when P_1 and P_2 have no common vertices but a and x. Then there was an even number of vertices in P_1 and odd in P_2 so together it is ODD (we counted a and x twice which does not change even/odd) and we get a contradiction.

If P_1 and P_2 have common vertices other then a and x say $q_1, \dots q_k$ then



Notice that we can compute number of vertices in $P_1 \cup P_2$ in two ways:

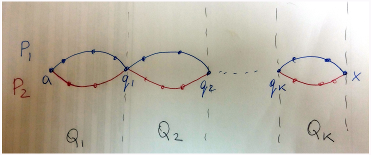
$$\#P_1 + \#P_2 - 2 - k = \sum_{i=1}^{k} \#Q_i - k.$$

Thus $\#P_1 + \#P_2 - 2 = \sum_{i=1}^{k} \#Q_i$

Theorem:

A graph G is bipartite is and only if any circuit in G has even length.

Let's start our construction/algorithm. Pick up any vertex, say a, and put it in L. Now we must put all vertices adjacent to a in R. Put all vertices which are two edges away from a to L, so in general if a vertex is odd steps (odd number of edges) from a put it into R if even number of edges from a put it into L.
One problem with this algorithm is why it is well defined? i.e. why there is no vertex x such that there is pass P_1 of even length from a to x and pass P_2 of odd length from a to x. Assume (towards a contradiction) that we actually have such a situation, then consider $P_1 \cup P_2$ it is a circuit and we can "play" with it.
First, consider a case when P_1 and P_2 have no common vertices but a and x. Then there was an even number of vertices in P_1 and odd in P_2 so together it is ODD (we counted a and x twice which does not change even/odd) and we get a contradiction.
If P_1 and P_2 have common vertices other then a and x say $q_1, \ldots q_k$ then



Notice that we can compute number of vertices in $P_1 \cup P_2$ in two ways:

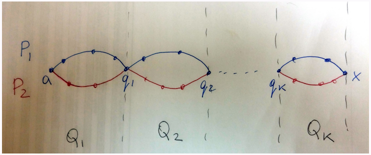
$$\#P_1 + \#P_2 - 2 - k = \sum_{i=1}^{k} \#Q_i - k.$$

Thus $\#P_1 + \#P_2 - 2 = \sum_{i=1}^{k} \#Q_i$ and $\sum_{i=1}^{k} \#Q_i$ is odd

Theorem:

A graph G is bipartite is and only if any circuit in G has even length.

Let's start our construction/algorithm. Pick up any vertex, say a, and put it in L. Now we must put all vertices adjacent to a in R. Put all vertices which are two edges away from a to L, so in general if a vertex is odd steps (odd number of edges) from a put it into R if even number of edges from a put it into L.
One problem with this algorithm is why it is well defined? i.e. why there is no vertex x such that there is pass P_1 of even length from a to x and pass P_2 of odd length from a to x. Assume (towards a contradiction) that we actually have such a situation, then consider $P_1 \cup P_2$ it is a circuit and we can "play" with it.
First, consider a case when P_1 and P_2 have no common vertices but a and x. Then there was an even number of vertices in P_1 and odd in P_2 so together it is ODD (we counted a and x twice which does not change even/odd) and we get a contradiction.
If P_1 and P_2 have common vertices other then a and x say $q_1, \ldots q_k$ then



Notice that we can compute number of vertices in $P_1 \cup P_2$ in two ways:

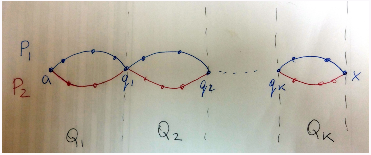
$$\#P_1 + \#P_2 - 2 - k = \sum_{i=1}^{k} \#Q_i - k.$$

Thus $\#P_1 + \#P_2 - 2 = \sum_{i=1}^{k} \#Q_i$ and $\sum_{i=1}^{k} \#Q_i$ is odd and THUS we must have at least one $\#Q_i$ be an odd number.

Theorem:

A graph G is bipartite is and only if any circuit in G has even length.

Let's start our construction/algorithm. Pick up any vertex, say a, and put it in L. Now we must put all vertices adjacent to a in R. Put all vertices which are two edges away from a to L, so in general if a vertex is odd steps (odd number of edges) from a put it into R if even number of edges from a put it into L.
One problem with this algorithm is why it is well defined? i.e. why there is no vertex x such that there is pass P_1 of even length from a to x and pass P_2 of odd length from a to x. Assume (towards a contradiction) that we actually have such a situation, then consider $P_1 \cup P_2$ it is a circuit and we can "play" with it.
First, consider a case when P_1 and P_2 have no common vertices but a and x. Then there was an even number of vertices in P_1 and odd in P_2 so together it is ODD (we counted a and x twice which does not change even/odd) and we get a contradiction.
If P_1 and P_2 have common vertices other then a and x say $q_1, \dots q_k$ then



Notice that we can compute number of vertices in $P_1 \cup P_2$ in two ways:

$$\#P_1 + \#P_2 - 2 - k = \sum_{i=1}^{k} \#Q_i - k.$$

Thus $\#P_1 + \#P_2 - 2 = \sum_{i=1}^{k} \#Q_i$ and $\sum_{i=1}^{k} \#Q_i$ is odd and THUS we must have at least one $\#Q_i$ be an odd number. So we found an odd length circuit which contradicts the assumption of

Theorem:

A graph G is bipartite is and only if any circuit in G has even length.

Let's start our construction. Pick up any vertex, say a, and put it in L. Now we must put all vertices adjacent to a in R. Put all vertices which are two edges away from a to L, so in general if a vertex is odd steps (odd number of edges) from a put it into R if even number of edges from a put it into L.

Theorem:

A graph G is bipartite is and only if any circuit in G has even length.

Let's start our construction. Pick up any vertex, say a, and put it in L. Now we must put all vertices adjacent to a in R. Put all vertices which are two edges away from a to L, so in general if a vertex is odd steps (odd number of edges) from a put it into R if even number of edges from a put it into L.

We still have one more additional issue with our algorithm we must show that there are no edges "inside" set L (or R),

Theorem:

A graph G is bipartite is and only if any circuit in G has even length.

Let's start our construction. Pick up any vertex, say a, and put it in L. Now we must put all vertices adjacent to a in R. Put all vertices which are two edges away from a to L, so in general if a vertex is odd steps (odd number of edges) from a put it into R if even number of edges from a put it into L.

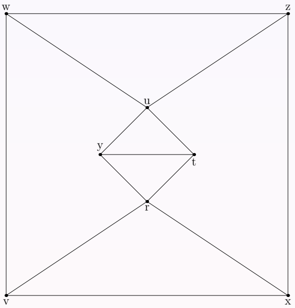
We still have one more additional issue with our algorithm we must show that there are no edges "inside" set L (or R), the idea is very similar to the previous discussion and we will leave it as a Home Work.

□

Theorem:

A graph G is bipartite is and only if any circuit in G has even length.

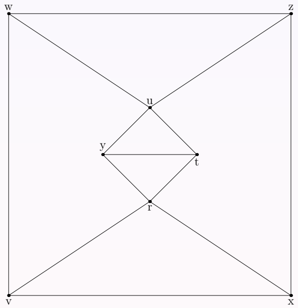
Consider a graph below, is it Bipartite?



Theorem:

A graph G is bipartite is and only if any circuit in G has even length.

Consider a graph below, is it Bipartite?



The answer is NO. We notice that there is an odd length circuit (y, u, t) and apply the theorem!