Lecture 4
MATH-42021/52021 Graph Theory and Combinatorics.

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June, 2016.

Planar Graph

One of the most natural examples of the graphs is street maps - which are planar, so it is interesting to define what is it "planar" and create a theory which will help us to understand is a given graph planar or not

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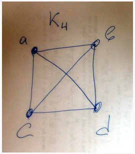

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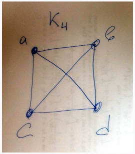
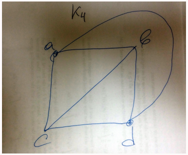


Clearly, as it is drawn it is not a planar graph, but K_4 is planar! Because we can draw it differently (and now as a planar graph):



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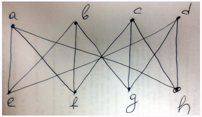
But for many graphs the task is not trivial at all how to determine is it planar or not, for example (we will figure it out in a few slides) this one:



Planar Graph: Interesting Application

One of the oldest problems in graph theory is connected with map coloring. The question is what is the minimal number of different colors are needed to color countries on some map so that any pair of countries with a common border are given different colors.
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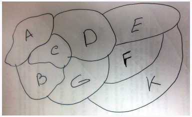

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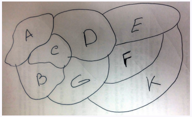

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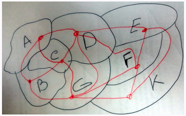

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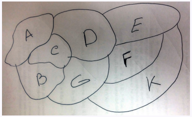

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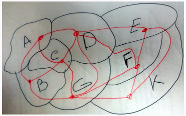

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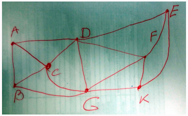

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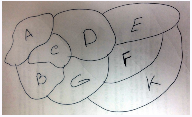

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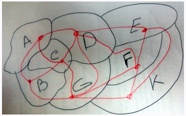

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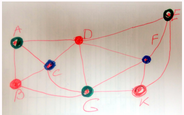

The question now is how many colors we need to "color" the vertices such that adjacent vertices have different colors. For the above graph we can do with 3 colors. Is it enough for any planar graph?

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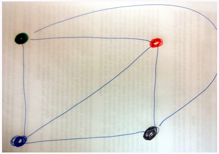

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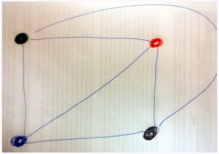

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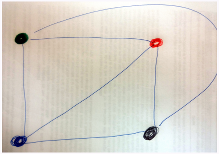

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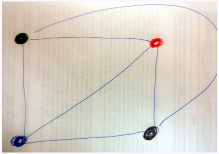

So we need at least 4 colors to color a planar graph (and thus a map). But would 4 colors be enough? The answer is YES, but this is a VERY non-trivial question which took a long time to be solved. But helped to develop a very interesting theory of planar graphs.

Planar Graph: How to draw - the circle-chord method.

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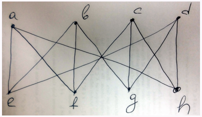

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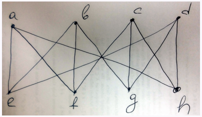

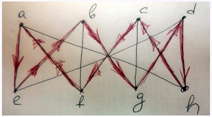

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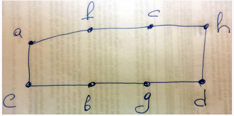

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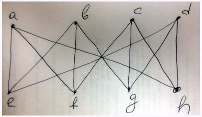

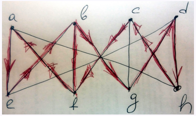

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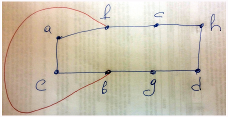

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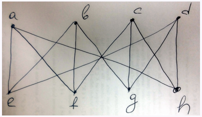

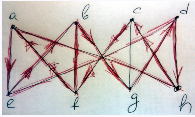

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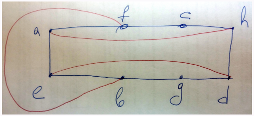

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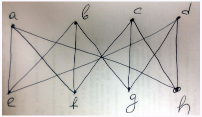

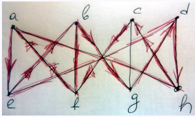

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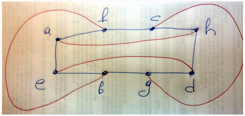

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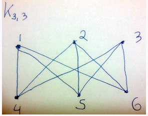

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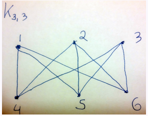

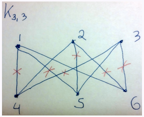

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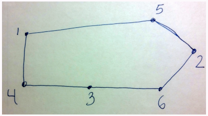

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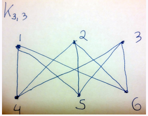

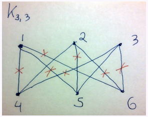

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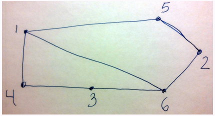

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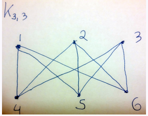

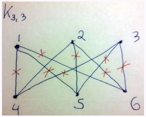

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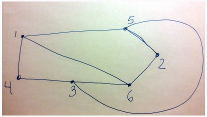

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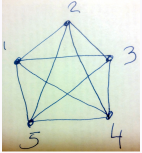
Let's play with yet another crucial example K_5:



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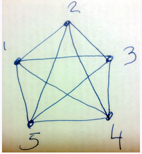
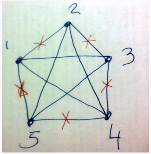
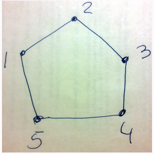




here we immediately see the required circuit:



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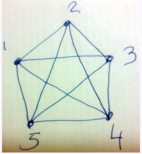

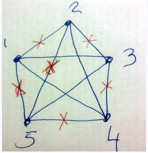

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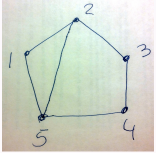

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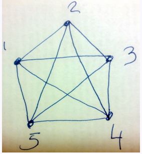

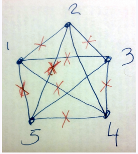

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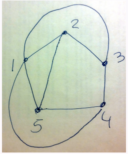

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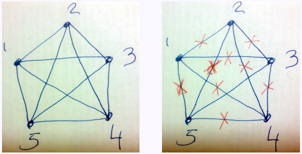

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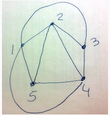

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Planar Graph: How to draw - the circle-chord method.

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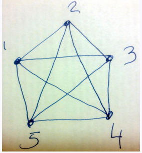

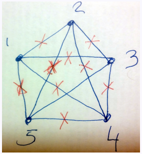

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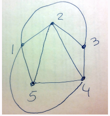

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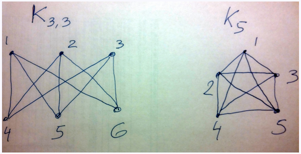
NOT Planar Graphs

We just proved that graphs $K_{3,3}$ and K_5 are not planar



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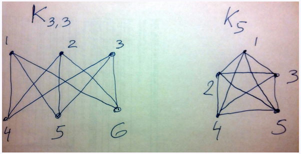

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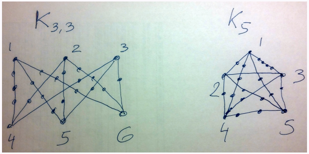

The above graphs are called **configurations** of $K_{3,3}$ and K_5.

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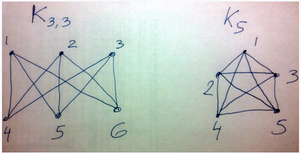

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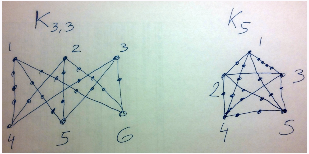

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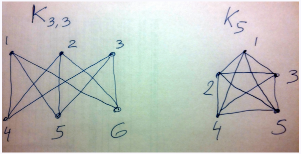

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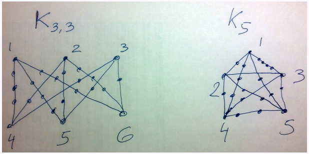

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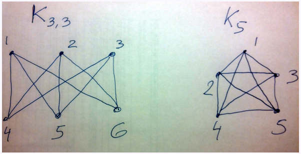

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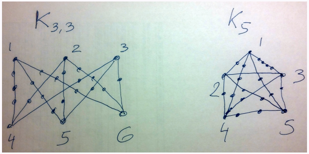

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Theorem (Kuratowski, 1930)

A graph is planar if and only if it does not contain a subgraph that is a K_5 or $K_{3,3}$ configuration.