Lecture 4
MATH-42021/52021 Graph Theory and Combinatorics.

Artem Zvavitch

Department of Mathematical Sciences, Kent State University

July, 2018.

Planar Graph

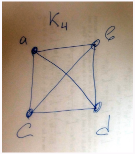
One of the most natural examples of the graphs is street maps - which are planar, so it is interesting to define what is it "planar" and create a theory which will help us to understand is a given graph planar or not

We say that a graph is **planar** if it can be drawn on a plane without edges crossing. We use the term **planar graph** to refer to a planar depiction of a "planar" graph.

One of the most natural examples of the graphs is street maps - which are planar, so it is interesting to define what is it "planar" and create a theory which will help us to understand is a given graph planar or not

We say that a graph is **planar** if it can be drawn on a plane without edges crossing. We use the term **planar graph** to refer to a planar depiction of a "planar" graph.

So what about :

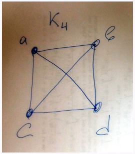


Clearly, as it is drawn it is not a planar graph,

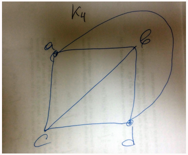
One of the most natural examples of the graphs is street maps - which are planar, so it is interesting to define what is it "planar" and create a theory which will help us to understand is a given graph planar or not

We say that a graph is **planar** if it can be drawn on a plane without edges crossing. We use the term **planar graph** to refer to a planar depiction of a "planar" graph.

So what about :



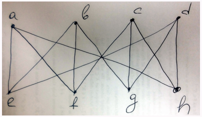
Clearly, as it is drawn it is not a planar graph, but K_4 is planar! Because we can draw it differently (and now as a planar graph):



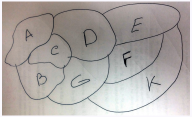
One of the most natural examples of the graphs is street maps - which are planar, so it is interesting to define what is it "planar" and create a theory which will help us to understand is a given graph planar or not

We say that a graph is **planar** if it can be drawn on a plane without edges crossing. We use the term **planar graph** to refer to a planar depiction of a "planar" graph.

But for many graphs the task is not trivial at all how to determine is it planar or not, for example (we will figure it out in a few slides) this one:

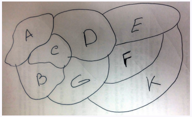


One of the oldest problems in graph theory is connected with map coloring. The question is what is the minimal number of different colors are needed to color countries on some map so that any pair of countries with a common border are given different colors.
Here an example (which is planar):

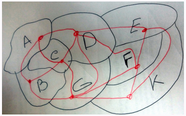


Planar Graph: Interesting Application

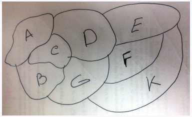
One of the oldest problems in graph theory is connected with map coloring. The question is what is the minimal number of different colors are needed to color countries on some map so that any pair of countries with a common border are given different colors.
Here an example (which is planar):



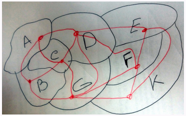
A closely related notion is a **dual graph** of the map which is more useful (vertices -> countries, put an edge if they share a border):



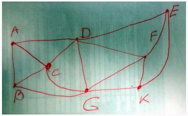
One of the oldest problems in graph theory is connected with map coloring. The question is what is the minimal number of different colors are needed to color countries on some map so that any pair of countries with a common border are given different colors.
Here an example (which is planar):



A closely related notion is a **dual graph** of the map which is more useful (vertices -> countries, put an edge if they share a border):



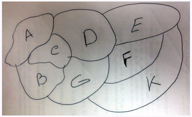
Or if we now draw it without "map":



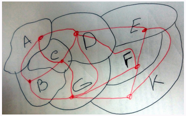
The question now is how many colors we need to "color" the vertices such that adjacent vertices have different colors.

Planar Graph: Interesting Application

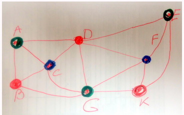
One of the oldest problems in graph theory is connected with map coloring. The question is what is the minimal number of different colors are needed to color countries on some map so that any pair of countries with a common border are given different colors.
Here an example (which is planar):



But a closely related notion is a **dual graph** of the map which is more useful (vertices -> countries, put an edge if they share a border):



Or if we now draw it without "map":

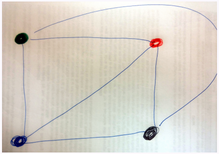


The question now is how many colors we need to "color" the vertices such that adjacent vertices have different colors. For the above graph we can do with 3 colors. Is it enough for any planar graph?

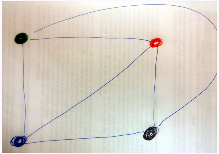
Planar Graph: Coloring

Would 3 colors be enough to color a planar graph?

Would 3 colors be enough to color a planar graph? NO $-K_4$ -all 4 vertices connected to each other so there is no way to color them in 3 colors

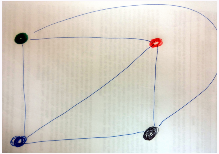


Would 3 colors be enough to color a planar graph? NO $-K_4$ -all 4 vertices connected to each other so there is no way to color them in 3 colors



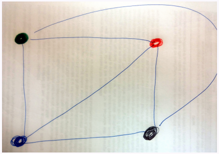
So we need at least 4 colors to color a planar graph (and thus a map).

Would 3 colors be enough to color a planar graph? NO $-K_4$ -all 4 vertices connected to each other so there is no way to color them in 3 colors



So we need at least 4 colors to color a planar graph (and thus a map). But would 4 colors be enough?

Would 3 colors be enough to color a planar graph? NO $-K_4$ -all 4 vertices connected to each other so there is no way to color them in 3 colors



So we need at least 4 colors to color a planar graph (and thus a map). But would 4 colors be enough? The answer is YES, but this is a VERY non-trivial question which took a long time to be solved. But helped to develop a very interesting theory of planar graphs.

Planar Graph: How to draw - the circle-chord method.

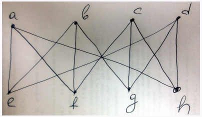
We would like to create a method/algorithm which would help us to draw a given graph on the plane.

We would like to create a method/algorithm which would help us to draw a given graph on the plane. We would like also be able to say that if our algorithm fails then the graph is not planar.

Planar Graph: How to draw - the circle-chord method.

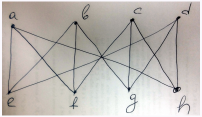
We would like to create a method/algorithm which would help us to draw a given graph on the plane. We would like also be able to say that if our algorithm fails then the graph is not planar. The idea we propose will NOT work for all graphs (but will for the most we will play with) and comes from the following observation.

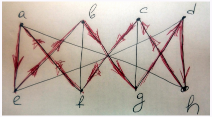
We would like to create a method/algorithm which would help us to draw a given graph on the plane. We would like also be able to say that if our algorithm fails then the graph is not planar. The idea we propose will NOT work for all graphs (but will for the most we will play with) and comes from the following observation. If we are lucky and if a graph G contains a circuit such that all vertices belong to this circuit then it is clear how to start - draw this circuit, there is no much choice. For example consider the graph we observed a few slides ago



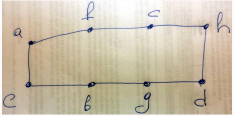
Just go around the graph and create a circuit (IF POSSIBLE) starting from vertex a (arrows is only to help you to see how we guessed for the circuit)

We would like to create a method/algorithm which would help us to draw a given graph on the plane. We would like also be able to say that if our algorithm fails then the graph is not planar. The idea we propose will NOT work for all graphs (but will for the most we will play with) and comes from the following observation. If we are lucky and if a graph G contains a circuit such that all vertices belong to this circuit then it is clear how to start - draw this circuit, there is no much choice. For example consider the graph we observed a few slides ago





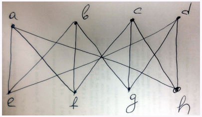
Just go around the graph and create a circuit (IF POSSIBLE) starting from vertex a (arrows are only to help you to see how we guessed for the circuit). Here a circuit we got:

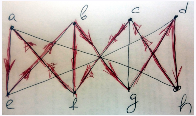


Now the main question where to put the first edge we have not yet used -> INSIDE OR OUTSIDE circuit? actually, this does not meter! Notice that actually "Inside" and "outside" of the circuit now symmetric to each other indeed we can always turn our final drawing turn inside out with respect to circuit.

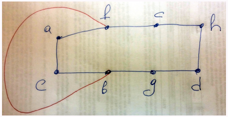
Planar Graph: How to draw - the circle-chord method.

We would like to create a method/algorithm which would help us to draw a given graph on the plane. We would like also be able to say that if our algorithm fails then the graph is not planar. The idea we propose will NOT work for all graphs (but will for the most we will play with) and comes from the following observation. If we are lucky and if a graph G contains a circuit such that all vertices belong to this circuit then it is clear how to start - draw this circuit, there is no much choice. For example consider the graph we observed a few slides ago





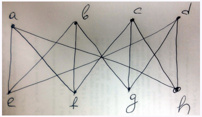
Just go around the graph and create a circuit (IF POSSIBLE) starting from vertex a (arrows are only to help you to see how we guessed for the circuit). Here a circuit we got:

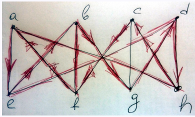


Now the main question where to put the first edge we have not yet used -> INSIDE OR OUTSIDE circuit? actually, this does not meter! Notice that actually "Inside" and "outside" of the circuit now symmetric to each other indeed we can always turn our final drawing turn inside out with respect to circuit. So we choose to put is outside. Say we draw edge (b, f) and we mark it on original graph as done. Next we need to consider vertices a and e notice there is no much choice for edges incident to them (they can not cross (f, b) thus must be inside original circuit).

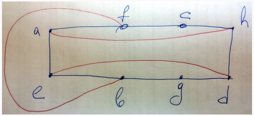
Planar Graph: How to draw - the circle-chord method.

We would like to create a method/algorithm which would help us to draw a given graph on the plane. We would like also be able to say that if our algorithm fails then the graph is not planar. The idea we propose will NOT work for all graphs (but will for the most we will play with) and comes from the following observation. If we are lucky and if a graph G contains a circuit such that all vertices belong to this circuit then it is clear how to start - draw this circuit, there is no much choice. For example consider the graph we observed a few slides ago





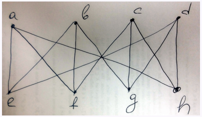
Just go around the graph and create a circuit (IF POSSIBLE) starting from vertex a (arrows are only to help you to see how we guessed for the circuit). Here a circuit we got:

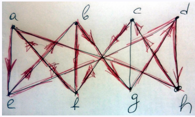


Now the main question where to put the first edge we have not yet used -> INSIDE OR OUTSIDE circuit? actually, this does not meter! Notice that actually "Inside" and "outside" of the circuit now symmetric to each other indeed we can always turn our final drawing turn inside out with respect to circuit. So we choose to put is outside. Say we draw edge (b, f) and we mark it on original graph as done. Next we need to consider vertices a and e notice there is no much choice for edges incident to them (they can not cross (f, b) thus must be inside original circuit). We mark them, again, on original graph as used. Finally we see that there is no much choice but to put (c, g) outside and ....

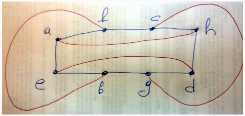
Planar Graph: How to draw - the circle-chord method.

We would like to create a method/algorithm which would help us to draw a given graph on the plane. We would like also be able to say that if our algorithm fails then the graph is not planar. The idea we propose will NOT work for all graphs (but will for the most we will play with) and comes from the following observation. If we are lucky and if a graph G contains a circuit such that all vertices belong to this circuit then it is clear how to start - draw this circuit, there is no much choice. For example consider the graph we observed a few slides ago



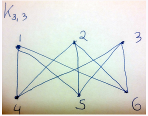


Just go around the graph and create a circuit (IF POSSIBLE) starting from vertex a (arrows are only to help you to see how we guessed for the circuit). Here a circuit we got:



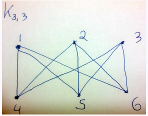
Now the main question where to put the first edge we have not yet used -> INSIDE OR OUTSIDE circuit? actually, this does not meter! Notice that actually "Inside" and "outside" of the circuit now symmetric to each other indeed we can always turn our final drawing turn inside out with respect to circuit. So we choose to put is outside. Say we draw edge (b, f) and we mark it on original graph as done. Next we need to consider vertices a and e notice there is no much choice for edges incident to them (they can not cross (f, b) thus must be inside original circuit). We mark them, again, on original graph as used. Finally we see that there is no much choice but to put (c, g) outside and to celebrate – the graph is planar!

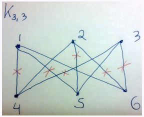
Let's play with another (this time crucial!) example $K_{3,3}$ (a bipartite graph build on two sets of vertices of size 3 each, when you provide all possible connections between two sets):



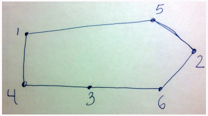
Now we can try to find a circuit which contains all vertices:

Let's play with another (this time crucial!) example $K_{3,3}$ (a bipartite graph build on two sets of vertices of size 3 each, when you provide all possible connections between two sets):

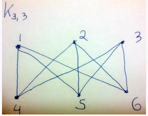


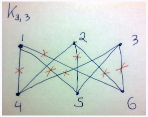


Now we can try to find a circuit which contains all vertices (and mark the edges we used):

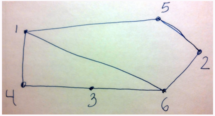


Let's play with another (this time crucial!) example $K_{3,3}$ (a bipartite graph build on two sets of vertices of size 3 each, when you provide all possible connections between two sets):





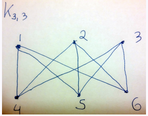
Now we can try to find a circuit which contains all vertices (and mark the edges we used):

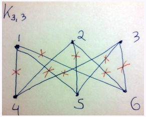


Next, we select an unused edge and put it inside or outside (remember "inside out " trick) so for example pic us $(1,6)$ and put it inside and mark as used.

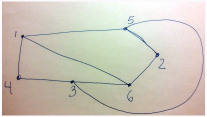
Planar Graph: How to draw - the circle-chord method.

Let's play with another (this time crucial!) example $K_{3,3}$ (a bipartite graph build on two sets of vertices of size 3 each, when you provide all possible connections between two sets):





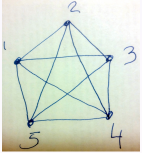
Now we can try to find a circuit which contains all vertices (and mark the edges we used):



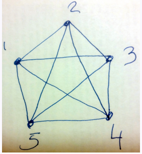
Next, we select an unused edge and put it inside or outside (remember "inside out " trick) so for example pic us $(1, 6)$ and put it inside and mark as used. Now notice that and edge $(3, 5)$ has no choice but to go outside. Next we left with and edge $(2, 4)$ it there is NO way to put is inside or outside without crossing and thus $K_{3,3}$ **is not planar!**

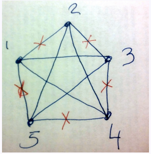
Planar Graph: How to draw - the circle-chord method.

Let's play with yet another crucial example K_5:

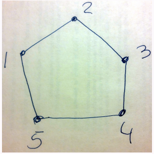


Let's play with yet another crucial example K_5:

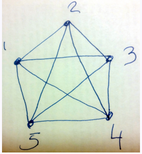


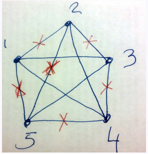


here we immediately see the required circuit:

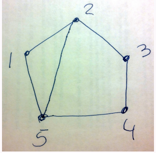


Let's play with yet another crucial example K_5:



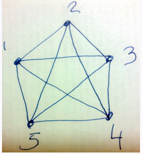


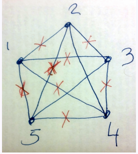
here we immediately see the required circuit:



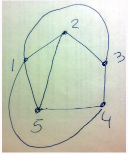
As before select and edge, for example $(2,5)$ and put it inside (do not forget to mark it as used!).

Let's play with yet another crucial example K_5:





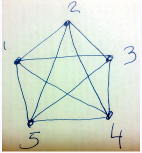
here we immediately see the required circuit:

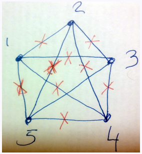


As before select and edge, for example $(2, 5)$ and put it inside (do not forget to mark it as used!). Next, notice that we have no much choice but to put unused edges with end at 1 outside.

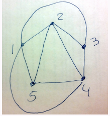
Planar Graph: How to draw - the circle-chord method.

Let's play with yet another crucial example K_5:





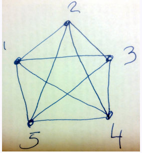
here we immediately see the required circuit:

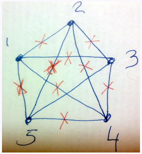


As before select and edge, for example $(2, 5)$ and put it inside (do not forget to mark it as used!). Next, notice that we have no much choice but to put unused edges with end at 1 outside. Edge $(2, 4)$ must go inside now. Finally we left with and edge $(3, 5)$.

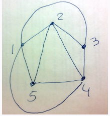
Planar Graph: How to draw - the circle-chord method.

Let's play with yet another crucial example K_5:





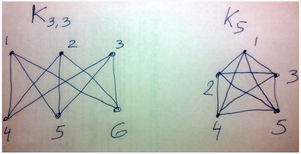
here we immediately see the required circuit:



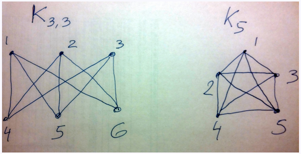
As before select and edge, for example $(2, 5)$ and put it inside (do not forget to mark it as used!). Next, notice that we have no much choice but to put unused edges with end at 1 outside. Edge $(2, 4)$ must go inside now. Finally we left with and edge $(3, 5)$. There is NO way to put is inside or outside without crossing and thus K_5 **is not planar!**

NOT Planar Graphs

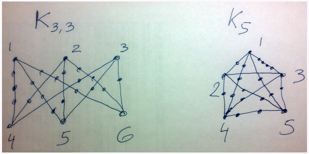
We just proved that graphs $K_{3,3}$ and K_5 are not planar



We just proved that graphs $K_{3,3}$ and K_5 are not planar

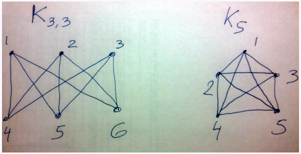


Notice that if we simply add more vertices ON EDGES of the graphs this would not make them planar (careful! do not put vertices on "intersection" of edges in plane drawing of the graph).

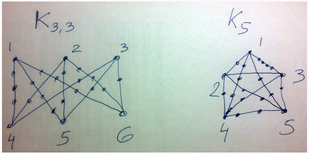


The above graphs are called **configurations** of $K_{3,3}$ and K_5.

We just proved that graphs $K_{3,3}$ and K_5 are not planar



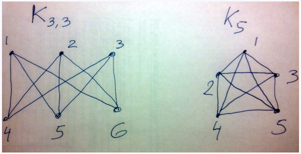
Notice that if we simply add more vertices ON EDGES of the graphs this would not make them planar (careful! do not put vertices on "intersection" of edges in plane drawing of the graph).



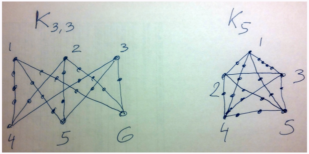
The above graphs are called **configurations** of $K_{3,3}$ and K_5. It is not so hard to guess that if a subgraph of the graph is non planar then the graph itself must be non planar.

NOT Planar Graphs

We just proved that graphs $K_{3,3}$ and K_5 are not planar

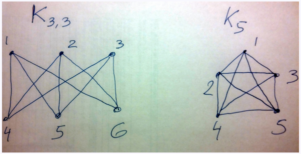


Notice that if we simply add more vertices ON EDGES of the graphs this would not make them planar (careful! do not put vertices on "intersection" of edges in plane drawing of the graph).

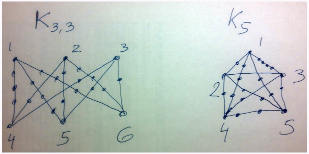


The above graphs are called **configurations** of $K_{3,3}$ and K_5. It is not so hard to guess that if a subgraph of the graph is non planar then the graph itself must be non planar. Thus if G contains a configuration of $K_{3,3}$ or K_5 then it is non-planar.

We just proved that graphs $K_{3,3}$ and K_5 are not planar



Notice that if we simply add more vertices ON EDGES of the graphs this would not make them planar (careful! do not put vertices on "intersection" of edges in plane drawing of the graph).



The above graphs are called **configurations** of $K_{3,3}$ and K_5. It is not so hard to guess that if a subgraph of the graph is non planar then the graph itself must be non planar. Thus if G contains a configuration of $K_{3,3}$ or K_5 then it is non-planar. The amazing thing that this fact characterize planar graphs:

Theorem (Kuratowski, 1930)

A graph is planar if and only if it does not contain a subgraph that is a K_5 or $K_{3,3}$ configuration.