Lecture 5
MATH-42021/52021 Graph Theory and Combinatorics.

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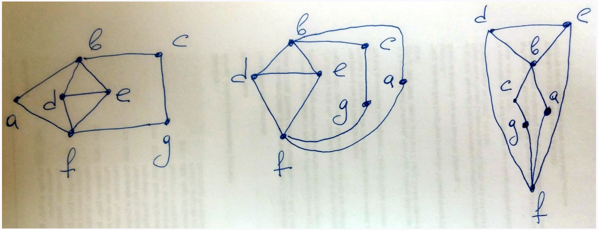
Department of Mathematical Sciences, Kent State University

June, 2016.

Planar Graph - continuation of the story.

There may be a lot of ways to draw a planar graph:



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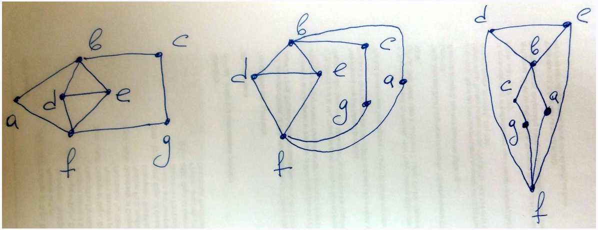

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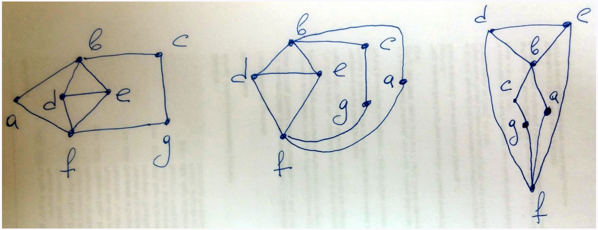

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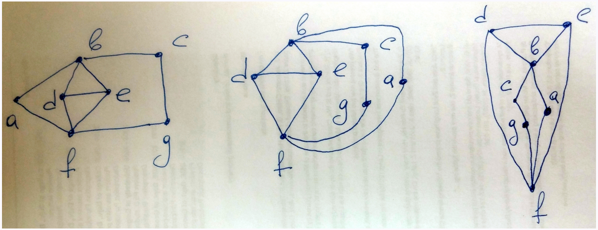

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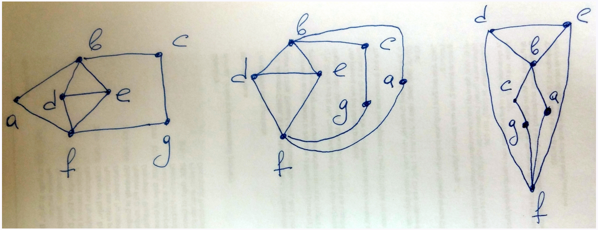

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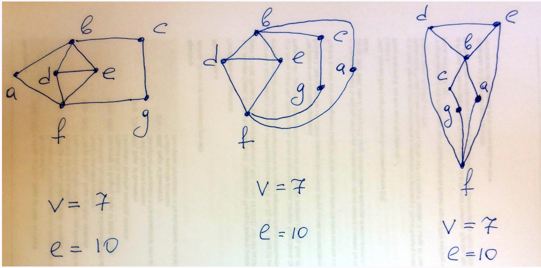

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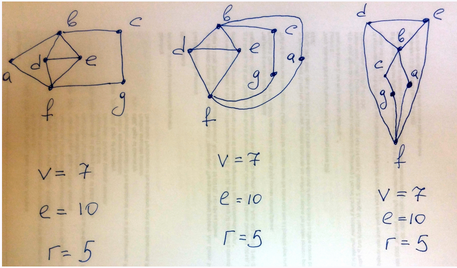

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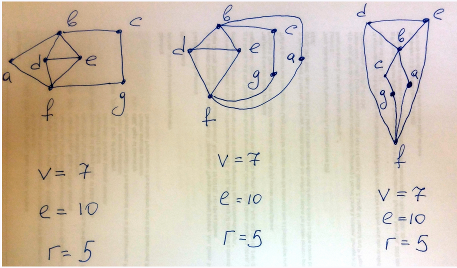

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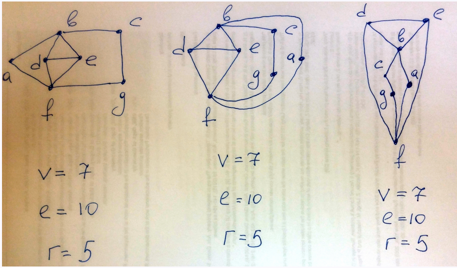

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Theorem: Euler's Formula

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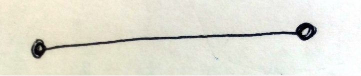

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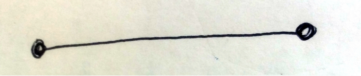

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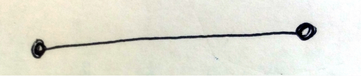

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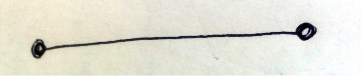

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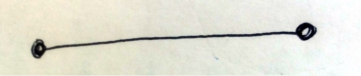

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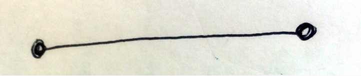

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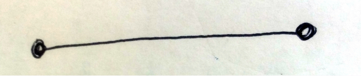

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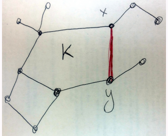

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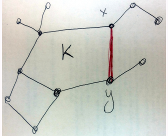

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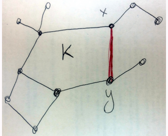

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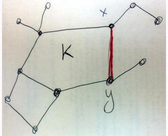

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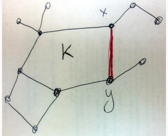

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$$(\boldsymbol{r}_{n+1} - 1) = (\boldsymbol{e}_{n+1} - 1) - \boldsymbol{v}_{n+1} + 2.$$

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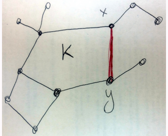

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Cancelling -1 from both sides we get the formula we need for G_{n+1}. YES!!!

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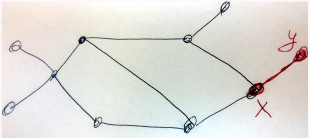

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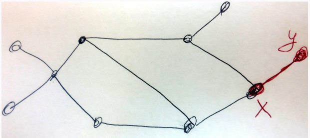

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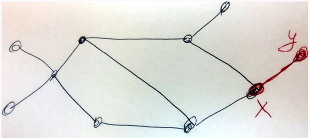

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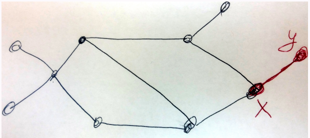

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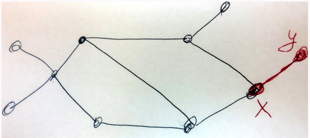

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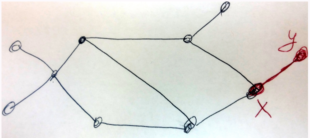

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Cancelling $+1$ on the right we get the formula we need for G_{n+1}.

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$$\boldsymbol{r} = \boldsymbol{e} - \boldsymbol{v} + 2 = 15 - 10 + 2 = 7.$$

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Euler's Formula – Corollary

It would be great to use Euler's formula to check if a given graph is planar, indeed, IF the graph does not satisfy the formula it is not planar. The problem is "number of regions" –when we are given the graph and before we drawn it on the plane it may be impossible to find $\boldsymbol{r}$, especially if it does not exists!

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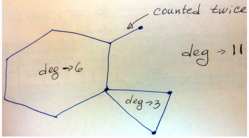

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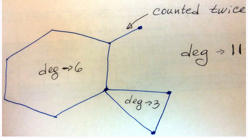

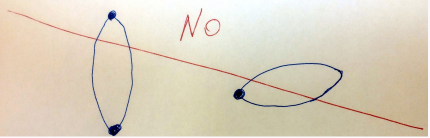

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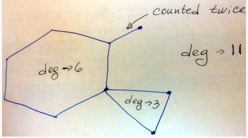

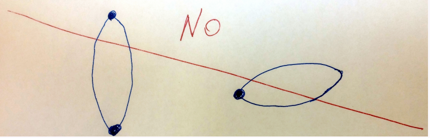

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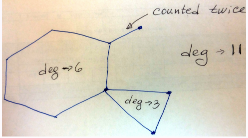

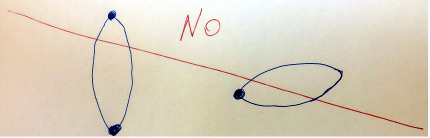

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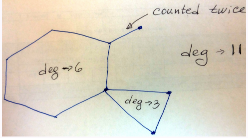

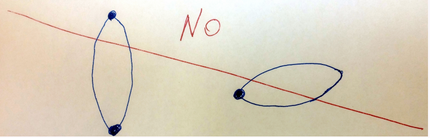

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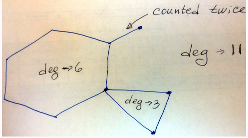

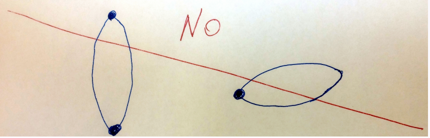

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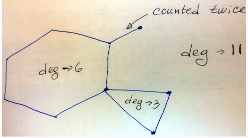

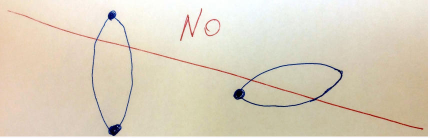

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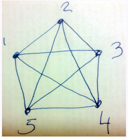

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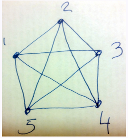

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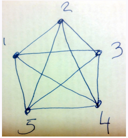

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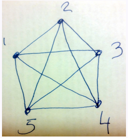

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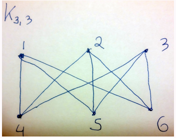

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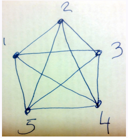

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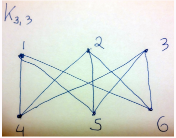

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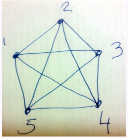

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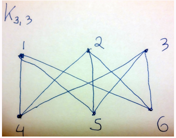

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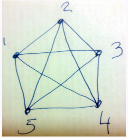

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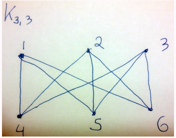

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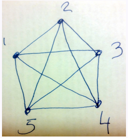

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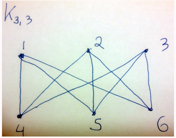

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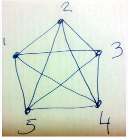

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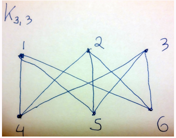

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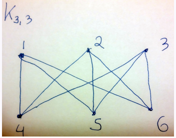

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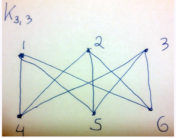

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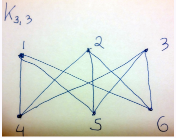

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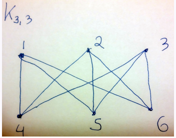

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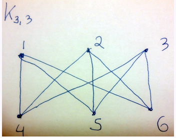

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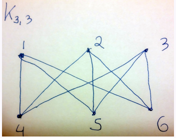

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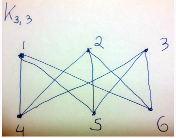

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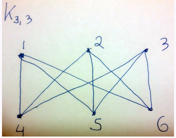

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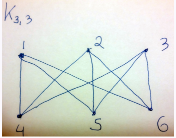

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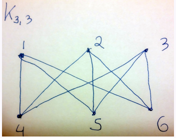

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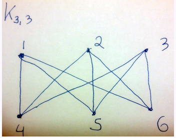
Just to show that Euler's formula is very strong let us see how we can still use it to solve $K_{3,3}$ problem. Notice that $K_{3,3}$ is a bipartite graph. We proved that those guys may have circuits of even length only! Thus a shortest circuit we may have is of length 4 (and NOT 3 as for a general graph!). In the proof of the corollary we computed the sum of all degrees of regions (and the estimate we got was greater then $3\boldsymbol{r}$), now we can improve this estimate for a bipartite graph – $4\boldsymbol{r}$ and get in this special case inequality $2\boldsymbol{e} \geq 4\boldsymbol{r}$ or $\frac{1}{2}\boldsymbol{e} \geq \boldsymbol{r}$. Plug it in Euler formula $\frac{1}{2}\boldsymbol{e} \geq \boldsymbol{r} = \boldsymbol{e} - \boldsymbol{v} + 2$ and get a **special case of the corollary for bipartite connected planar graphs**:

$$\boldsymbol{e} \leq 2\boldsymbol{v} - 4$$

Corollary to Euler's formula:

If G is a connected planar graph with $\boldsymbol{e} > 1$, then $\boldsymbol{e} \leq 3\boldsymbol{v} - 6$.

Lets apply it to our other old friend $K_{3,3}$:



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Now it is easy to check that $K_{3,3}$ does not satisfy it (indeed $9 \not\leq 2 * 6 - 4$) and thus it is not planar.