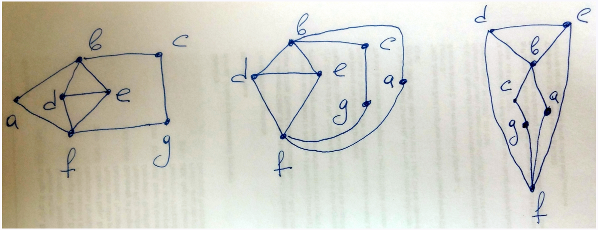
Lecture 5
MATH-42021/52021 Graph Theory and Combinatorics.

Artem Zvavitch

Department of Mathematical Sciences, Kent State University

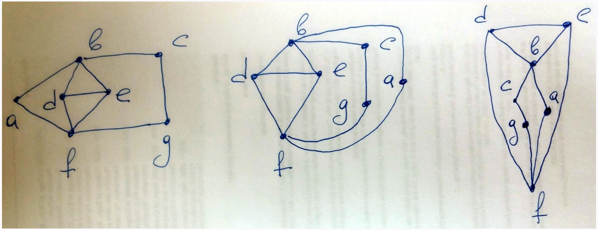
July, 2018.

There may be a lot of ways to draw a planar graph:



Planar Graph - continuation of the story.

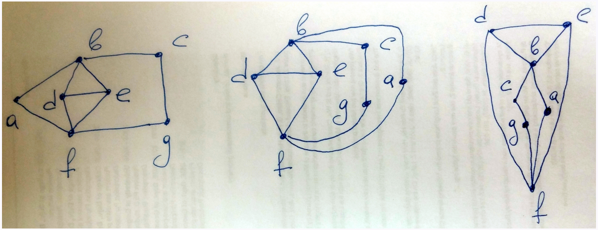
There may be a lot of ways to draw a planar graph:



Is there any pattern? connection?

Planar Graph - continuation of the story.

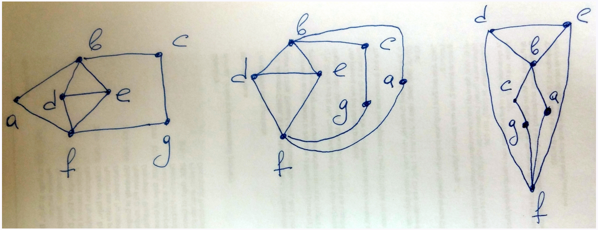
There may be a lot of ways to draw a planar graph:



Is there any pattern? connection? It turns out that the answer is yes. To stay the next theorem we need a couple of new definitions:

Planar Graph - continuation of the story.

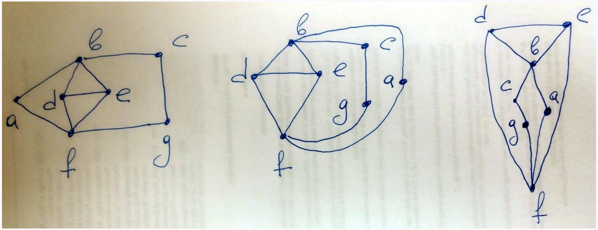
There may be a lot of ways to draw a planar graph:



Is there any pattern? connection? It turns out that the answer is yes. To stay the next theorem we need a couple of new definitions:

- **Connected** graph - for any pair of vertices there is path with end points at those vertices.

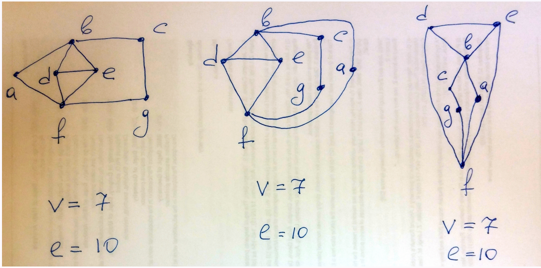
There may be a lot of ways to draw a planar graph:



Is there any pattern? connection? It turns out that the answer is yes. To stay the next theorem we need a couple of new definitions:

- **Connected** graph - for any pair of vertices there is path with end points at those vertices.
- ***v*** - number of vertices in the graph.
- ***e*** - number of edges in the graph.

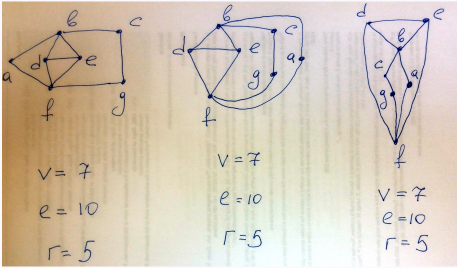
There may be a lot of ways to draw a planar graph:



Is there any pattern? connection? It turns out that the answer is yes. To stay the next theorem we need a couple of new definitions:

- **Connected** graph - for any pair of vertices there is path with end points at those vertices.
- v - number of vertices in the graph.
- e - number of edges in the graph.

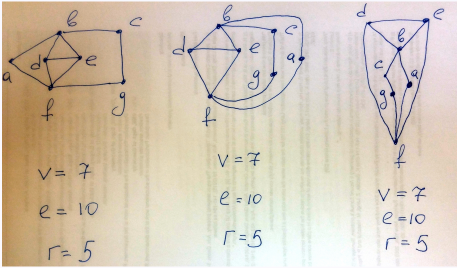
There may be a lot of ways to draw a planar graph:



Is there any pattern? connection? It turns out that the answer is yes. To stay the next theorem we need a couple of new definitions:

- **Connected** graph - for any pair of vertices there is path with end points at those vertices.
- $\boldsymbol{v}$ - number of vertices in the graph.
- $\boldsymbol{e}$ - number of edges in the graph.
- $\boldsymbol{r}$ - number of regions the planar graph defines on the plane (do not forget outside of the graph!).

There may be a lot of ways to draw a planar graph:



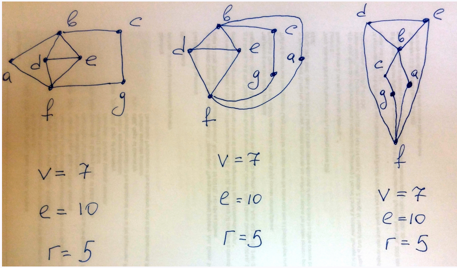
Is there any pattern? connection? It turns out that the answer is yes. To stay the next theorem we need a couple of new definitions:

- **Connected** graph - for any pair of vertices there is path with end points at those vertices.
- v - number of vertices in the graph.
- e - number of edges in the graph.
- r - number of regions the planar graph defines on the plane (do not forget outside of the graph!).

Amazing observation: r - does not change!

Planar Graph - continuation of the story.

There may be a lot of ways to draw a planar graph:



Is there any pattern? connection? It turns out that the answer is yes. To stay the next theorem we need a couple of new definitions:

- **Connected** graph - for any pair of vertices there is path with end points at those vertices.
- v - number of vertices in the graph.
- e - number of edges in the graph.
- r - number of regions the planar graph defines on the plane (do not forget outside of the graph!).

Amazing observation: r - does not change! Moreover,......

Theorem: Euler's Formula

If G is a connected planar graph, then any plane graph depiction of G has $\boldsymbol{r} = \boldsymbol{e} - \boldsymbol{v} + 2$ regions.

If G is a connected planar graph, then any plane graph depiction of G has $\boldsymbol{r} = \boldsymbol{e} - \boldsymbol{v} + 2$ regions.

Proof : We will use the method of mathematical induction to prove the theorem. The idea is that we will first prove the theorem for connected, planar graphs with just one edge (a starting point for induction, case "$n = 1$").

If G is a connected planar graph, then any plane graph depiction of G has $\boldsymbol{r} = \boldsymbol{e} - \boldsymbol{v} + 2$ regions.

Proof : We will use the method of mathematical induction to prove the theorem. The idea is that we will first prove the theorem for connected, planar graphs with just one edge (a starting point for induction, case "$n = 1$"). Next we will assume that the theorem is true for a connected, planar graph with n edges, where n is some fixed natural number

Theorem: Euler's Formula

If G is a connected planar graph, then any plane graph depiction of G has $\boldsymbol{r} = \boldsymbol{e} - \boldsymbol{v} + 2$ regions.

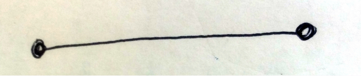
Proof : We will use the method of mathematical induction to prove the theorem. The idea is that we will first prove the theorem for connected, planar graphs with just one edge (a starting point for induction, case "$n = 1$"). Next we will assume that the theorem is true for a connected, planar graph with n edges, where n is some fixed natural number – and our goal will be to show that from this assumption we can always make the next step, i.e. to show that the theorem is true for a connected planar graph with $n + 1$ edges.

If G is a connected planar graph, then any plane graph depiction of G has $\boldsymbol{r} = \boldsymbol{e} - \boldsymbol{v} + 2$ regions.

Proof : We will use the method of mathematical induction to prove the theorem. The idea is that we will first prove the theorem for connected, planar graphs with just one edge (a starting point for induction, case "$n = 1$"). Next we will assume that the theorem is true for a connected, planar graph with n edges, where n is some fixed natural number – and our goal will be to show that from this assumption we can always make the next step, i.e. to show that the theorem is true for a connected planar graph with $n + 1$ edges.
Let G_1 will be a planar, connected graph with just one edge,

If G is a connected planar graph, then any plane graph depiction of G has $\boldsymbol{r} = \boldsymbol{e} - \boldsymbol{v} + 2$ regions.

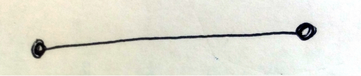
Proof : We will use the method of mathematical induction to prove the theorem. The idea is that we will first prove the theorem for connected, planar graphs with just one edge (a starting point for induction, case "$n = 1$"). Next we will assume that the theorem is true for a connected, planar graph with n edges, where n is some fixed natural number – and our goal will be to show that from this assumption we can always make the next step, i.e. to show that the theorem is true for a connected planar graph with $n + 1$ edges.
Let G_1 will be a planar, connected graph with just one edge, but then it must have exactly two vertices and defines just one region:



Denote by $\boldsymbol{v}_1, \boldsymbol{e}_1$ and $\boldsymbol{r}_1$ number of vertices, edges and regions of G_1.

If G is a connected planar graph, then any plane graph depiction of G has $\boldsymbol{r} = \boldsymbol{e} - \boldsymbol{v} + 2$ regions.

Proof : We will use the method of mathematical induction to prove the theorem. The idea is that we will first prove the theorem for connected, planar graphs with just one edge (a starting point for induction, case "$n = 1$"). Next we will assume that the theorem is true for a connected, planar graph with n edges, where n is some fixed natural number – and our goal will be to show that from this assumption we can always make the next step, i.e. to show that the theorem is true for a connected planar graph with $n + 1$ edges.
Let G_1 will be a planar, connected graph with just one edge, but then it must have exactly two vertices and defines just one region:

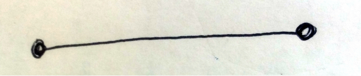


Denote by $\boldsymbol{v}_1, \boldsymbol{e}_1$ and $\boldsymbol{r}_1$ number of vertices, edges and regions of G_1. Then

$$\boldsymbol{v}_1 = 2, \boldsymbol{e}_1 = 1 \text{ and } \boldsymbol{r}_1 = 1$$

If G is a connected planar graph, then any plane graph depiction of G has $\boldsymbol{r} = \boldsymbol{e} - \boldsymbol{v} + 2$ regions.

Proof : We will use the method of mathematical induction to prove the theorem. The idea is that we will first prove the theorem for connected, planar graphs with just one edge (a starting point for induction, case "$n = 1$"). Next we will assume that the theorem is true for a connected, planar graph with n edges, where n is some fixed natural number – and our goal will be to show that from this assumption we can always make the next step, i.e. to show that the theorem is true for a connected planar graph with $n + 1$ edges.
Let G_1 will be a planar, connected graph with just one edge, but then it must have exactly two vertices and defines just one region:



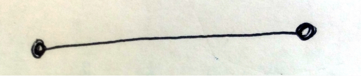
Denote by $\boldsymbol{v}_1, \boldsymbol{e}_1$ and $\boldsymbol{r}_1$ number of vertices, edges and regions of G_1. Then

$$\boldsymbol{v}_1 = 2, \boldsymbol{e}_1 = 1 \text{ and } \boldsymbol{r}_1 = 1$$

So $\boldsymbol{r}_1 = \boldsymbol{e}_1 - \boldsymbol{v}_1 + 2$, and the statement is true for $n = 1$!

If G is a connected planar graph, then any plane graph depiction of G has $\mathbf{r} = \mathbf{e} - \mathbf{v} + 2$ regions.

Proof : We will use the method of mathematical induction to prove the theorem. The idea is that we will first prove the theorem for connected, planar graphs with just one edge (a starting point for induction, case "$n = 1$"). Next we will assume that the theorem is true for a connected, planar graph with n edges, where n is some fixed natural number – and our goal will be to show that from this assumption we can always make the next step, i.e. to show that the theorem is true for a connected planar graph with $n+1$ edges.
Let G_1 will be a planar, connected graph with just one edge, but then it must have exactly two vertices and defines just one region:



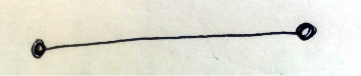
Denote by $\mathbf{v}_1, \mathbf{e}_1$ and $\mathbf{r}_1$ number of vertices, edges and regions of G_1. Then

$$\mathbf{v}_1 = 2, \mathbf{e}_1 = 1 \text{ and } \mathbf{r}_1 = 1$$

So $\mathbf{r}_1 = \mathbf{e}_1 - \mathbf{v}_1 + 2$, and the statement is true for $n = 1$!
Now, we assume that the theorem is true for any planar, connected graph of n-edges. i.e.

If G is a connected planar graph, then any plane graph depiction of G has $\boldsymbol{r} = \boldsymbol{e} - \boldsymbol{v} + 2$ regions.

Proof : We will use the method of mathematical induction to prove the theorem. The idea is that we will first prove the theorem for connected, planar graphs with just one edge (a starting point for induction, case "$n = 1$"). Next we will assume that the theorem is true for a connected, planar graph with n edges, where n is some fixed natural number – and our goal will be to show that from this assumption we can always make the next step, i.e. to show that the theorem is true for a connected planar graph with $n + 1$ edges.
Let G_1 will be a planar, connected graph with just one edge, but then it must have exactly two vertices and defines just one region:



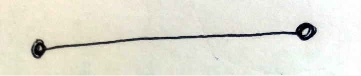
Denote by $\boldsymbol{v}_1, \boldsymbol{e}_1$ and $\boldsymbol{r}_1$ number of vertices, edges and regions of G_1. Then

$$\boldsymbol{v}_1 = 2, \boldsymbol{e}_1 = 1 \text{ and } \boldsymbol{r}_1 = 1$$

So $\boldsymbol{r}_1 = \boldsymbol{e}_1 - \boldsymbol{v}_1 + 2$, and the statement is true for $n = 1$!
Now, we assume that the theorem is true for any planar, connected graph of n-edges. i.e. if G_n is a planar, connected graph and $\boldsymbol{v}_n, \boldsymbol{e}_n$ and $\boldsymbol{r}_n$ number of vertices, edges and regions of G_n.

If G is a connected planar graph, then any plane graph depiction of G has $\boldsymbol{r} = \boldsymbol{e} - \boldsymbol{v} + 2$ regions.

Proof : We will use the method of mathematical induction to prove the theorem. The idea is that we will first prove the theorem for connected, planar graphs with just one edge (a starting point for induction, case "$n = 1$"). Next we will assume that the theorem is true for a connected, planar graph with n edges, where n is some fixed natural number – and our goal will be to show that from this assumption we can always make the next step, i.e. to show that the theorem is true for a connected planar graph with $n + 1$ edges.
Let G_1 will be a planar, connected graph with just one edge, but then it must have exactly two vertices and defines just one region:



Denote by $\boldsymbol{v}_1, \boldsymbol{e}_1$ and $\boldsymbol{r}_1$ number of vertices, edges and regions of G_1. Then

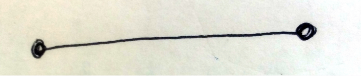
$$\boldsymbol{v}_1 = 2, \boldsymbol{e}_1 = 1 \text{ and } \boldsymbol{r}_1 = 1$$

So $\boldsymbol{r}_1 = \boldsymbol{e}_1 - \boldsymbol{v}_1 + 2$, and the statement is true for $n = 1$!
Now, we assume that the theorem is true for any planar, connected graph of n-edges. i.e. if G_n is a planar, connected graph and $\boldsymbol{v}_n, \boldsymbol{e}_n$ and $\boldsymbol{r}_n$ number of vertices, edges and regions of G_n. Then

$$\boldsymbol{r}_n = \boldsymbol{e}_n - \boldsymbol{v}_n + 2.$$

If G is a connected planar graph, then any plane graph depiction of G has $\boldsymbol{r} = \boldsymbol{e} - \boldsymbol{v} + 2$ regions.

Proof : We will use the method of mathematical induction to prove the theorem. The idea is that we will first prove the theorem for connected, planar graphs with just one edge (a starting point for induction, case "$n = 1$"). Next we will assume that the theorem is true for a connected, planar graph with n edges, where n is some fixed natural number – and our goal will be to show that from this assumption we can always make the next step, i.e. to show that the theorem is true for a connected planar graph with $n+1$ edges.
Let G_1 will be a planar, connected graph with just one edge, but then it must have exactly two vertices and defines just one region:



Denote by $\boldsymbol{v}_1, \boldsymbol{e}_1$ and $\boldsymbol{r}_1$ number of vertices, edges and regions of G_1. Then

$$\boldsymbol{v}_1 = 2, \boldsymbol{e}_1 = 1 \text{ and } \boldsymbol{r}_1 = 1$$

So $\boldsymbol{r}_1 = \boldsymbol{e}_1 - \boldsymbol{v}_1 + 2$, and the statement is true for $n = 1$!
Now, we assume that the theorem is true for any planar, connected graph of n-edges. i.e. if G_n is a planar, connected graph and $\boldsymbol{v}_n, \boldsymbol{e}_n$ and $\boldsymbol{r}_n$ number of vertices, edges and regions of G_n. Then

$$\boldsymbol{r}_n = \boldsymbol{e}_n - \boldsymbol{v}_n + 2.$$

Our goal now to show that we can make next step of induction and the theorem is true for any planar connected graph G_{n+1} of $n+1$ edges, i.e. $\boldsymbol{r}_{n+1} = \boldsymbol{e}_{n+1} - \boldsymbol{v}_{n+1} + 2$.

If G is a connected planar graph, then any plane graph depiction of G has $\boldsymbol{r} = \boldsymbol{e} - \boldsymbol{v} + 2$ regions.

Now, we assume that the theorem is true for any planar, connected graph of n-edges. i.e. if G_n is a planar, connected graph and $\boldsymbol{v}_n, \boldsymbol{e}_n$ and $\boldsymbol{r}_n$ number of vertices, edges and regions of G_n. Then

$$\boldsymbol{r}_n = \boldsymbol{e}_n - \boldsymbol{v}_n + 2.$$

Our goal now to show that we can make next step of induction and the theorem is true for any planar connected graph G_{n+1} of $n+1$ edges, i.e. $\boldsymbol{r}_{n+1} = \boldsymbol{e}_{n+1} - \boldsymbol{v}_{n+1} + 2$.

If G is a connected planar graph, then any plane graph depiction of G has $\boldsymbol{r} = \boldsymbol{e} - \boldsymbol{v} + 2$ regions.

Now, we assume that the theorem is true for any planar, connected graph of n-edges. i.e. if G_n is a planar, connected graph and $\boldsymbol{v}_n, \boldsymbol{e}_n$ and $\boldsymbol{r}_n$ number of vertices, edges and regions of G_n. Then

$$\boldsymbol{r}_n = \boldsymbol{e}_n - \boldsymbol{v}_n + 2.$$

Our goal now to show that we can make next step of induction and the theorem is true for any planar connected graph G_{n+1} of $n+1$ edges, i.e. $\boldsymbol{r}_{n+1} = \boldsymbol{e}_{n+1} - \boldsymbol{v}_{n+1} + 2$.
So let G_{n+1}, be planar connected graph of $n+1$ edges.

Theorem: Euler's Formula

If G is a connected planar graph, then any plane graph depiction of G has $\boldsymbol{r} = \boldsymbol{e} - \boldsymbol{v} + 2$ regions.

Now, we assume that the theorem is true for any planar, connected graph of n-edges. i.e. if G_n is a planar, connected graph and $\boldsymbol{v}_n, \boldsymbol{e}_n$ and $\boldsymbol{r}_n$ number of vertices, edges and regions of G_n. Then

$$\boldsymbol{r}_n = \boldsymbol{e}_n - \boldsymbol{v}_n + 2.$$

Our goal now to show that we can make next step of induction and the theorem is true for any planar connected graph G_{n+1} of $n+1$ edges, i.e. $\boldsymbol{r}_{n+1} = \boldsymbol{e}_{n+1} - \boldsymbol{v}_{n+1} + 2$.
So let G_{n+1}, be planar connected graph of $n+1$ edges. Assume that we are drawing G_{n+1} edge by edge on the plane in such a way that on each step we have a connected graph.

If G is a connected planar graph, then any plane graph depiction of G has $\boldsymbol{r} = \boldsymbol{e} - \boldsymbol{v} + 2$ regions.

Now, we assume that the theorem is true for any planar, connected graph of n-edges. i.e. if G_n is a planar, connected graph and $\boldsymbol{v}_n, \boldsymbol{e}_n$ and $\boldsymbol{r}_n$ number of vertices, edges and regions of G_n. Then

$$\boldsymbol{r}_n = \boldsymbol{e}_n - \boldsymbol{v}_n + 2.$$

Our goal now to show that we can make next step of induction and the theorem is true for any planar connected graph G_{n+1} of $n+1$ edges, i.e. $\boldsymbol{r}_{n+1} = \boldsymbol{e}_{n+1} - \boldsymbol{v}_{n+1} + 2$.
So let G_{n+1}, be planar connected graph of $n+1$ edges. Assume that we are drawing G_{n+1} edge by edge on the plane in such a way that on each step we have a connected graph. Let (x, y) be the last edge of G_{n+1} we draw (i.e. edge "number" $n+1$),

If G is a connected planar graph, then any plane graph depiction of G has $\boldsymbol{r} = \boldsymbol{e} - \boldsymbol{v} + 2$ regions.

Now, we assume that the theorem is true for any planar, connected graph of n-edges. i.e. if G_n is a planar, connected graph and $\boldsymbol{v}_n, \boldsymbol{e}_n$ and $\boldsymbol{r}_n$ number of vertices, edges and regions of G_n. Then

$$\boldsymbol{r}_n = \boldsymbol{e}_n - \boldsymbol{v}_n + 2.$$

Our goal now to show that we can make next step of induction and the theorem is true for any planar connected graph G_{n+1} of $n+1$ edges, i.e. $\boldsymbol{r}_{n+1} = \boldsymbol{e}_{n+1} - \boldsymbol{v}_{n+1} + 2$.
So let G_{n+1}, be planar connected graph of $n+1$ edges. Assume that we are drawing G_{n+1} edge by edge on the plane in such a way that on each step we have a connected graph. Let (x, y) be the last edge of G_{n+1} we draw (i.e. edge "number" $n+1$), the idea is to "remove" (x, y) and thus create a graph of just n edges and to use the assumption.

If G is a connected planar graph, then any plane graph depiction of G has $\boldsymbol{r} = \boldsymbol{e} - \boldsymbol{v} + 2$ regions.

Now, we assume that the theorem is true for any planar, connected graph of n-edges. i.e. if G_n is a planar, connected graph and $\boldsymbol{v}_n, \boldsymbol{e}_n$ and $\boldsymbol{r}_n$ number of vertices, edges and regions of G_n. Then

$$\boldsymbol{r}_n = \boldsymbol{e}_n - \boldsymbol{v}_n + 2.$$

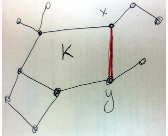
Our goal now to show that we can make next step of induction and the theorem is true for any planar connected graph G_{n+1} of $n+1$ edges, i.e. $\boldsymbol{r}_{n+1} = \boldsymbol{e}_{n+1} - \boldsymbol{v}_{n+1} + 2$.
So let G_{n+1}, be planar connected graph of $n+1$ edges. Assume that we are drawing G_{n+1} edge by edge on the plane in such a way that on each step we have a connected graph. Let (x, y) be the last edge of G_{n+1} we draw (i.e. edge "number" $n+1$), the idea is to "remove" (x, y) and thus create a graph of just n edges and to use the assumption. Note that when we remove (x, y) we will have planar graph (subgraph of planar is planar) and connected (this is how we were drawing G_{n+1}).

If G is a connected planar graph, then any plane graph depiction of G has $\boldsymbol{r} = \boldsymbol{e} - \boldsymbol{v} + 2$ regions.

Now, we assume that the theorem is true for any planar, connected graph of n-edges. i.e. if G_n is a planar, connected graph and $\boldsymbol{v}_n, \boldsymbol{e}_n$ and $\boldsymbol{r}_n$ number of vertices, edges and regions of G_n. Then

$$\boldsymbol{r}_n = \boldsymbol{e}_n - \boldsymbol{v}_n + 2.$$

Our goal now to show that we can make next step of induction and the theorem is true for any planar connected graph G_{n+1} of $n+1$ edges, i.e. $\boldsymbol{r}_{n+1} = \boldsymbol{e}_{n+1} - \boldsymbol{v}_{n+1} + 2$.
So let G_{n+1}, be planar connected graph of $n+1$ edges. Assume that we are drawing G_{n+1} edge by edge on the plane in such a way that on each step we have a connected graph. Let (x, y) be the last edge of G_{n+1} we draw (i.e. edge "number" $n+1$), the idea is to "remove" (x, y) and thus create a graph of just n edges and to use the assumption. Note that when we remove (x, y) we will have planar graph (subgraph of planar is planar) and connected (this is how we were drawing G_{n+1}). Still we will need to consider two cases (do we change regions or not):
In the first case x and y are were in G_n (the graph we draw before adding edge (x, y)):

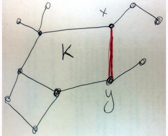


If G is a connected planar graph, then any plane graph depiction of G has $\boldsymbol{r} = \boldsymbol{e} - \boldsymbol{v} + 2$ regions.

Now, we assume that the theorem is true for any planar, connected graph of n-edges. i.e. if G_n is a planar, connected graph and $\boldsymbol{v}_n, \boldsymbol{e}_n$ and $\boldsymbol{r}_n$ number of vertices, edges and regions of G_n. Then

$$\boldsymbol{r}_n = \boldsymbol{e}_n - \boldsymbol{v}_n + 2.$$

Our goal now to show that we can make next step of induction and the theorem is true for any planar connected graph G_{n+1} of $n+1$ edges, i.e. $\boldsymbol{r}_{n+1} = \boldsymbol{e}_{n+1} - \boldsymbol{v}_{n+1} + 2$.
So let G_{n+1}, be planar connected graph of $n+1$ edges. Assume that we are drawing G_{n+1} edge by edge on the plane in such a way that on each step we have a connected graph. Let (x, y) be the last edge of G_{n+1} we draw (i.e. edge "number" $n+1$), the idea is to "remove" (x, y) and thus create a graph of just n edges and to use the assumption. Note that when we remove (x, y) we will have planar graph (subgraph of planar is planar) and connected (this is how we were drawing G_{n+1}). Still we will need to consider two cases (do we change regions or not):
In the first case x and y are were in G_n (the graph we draw before adding edge (x, y)):



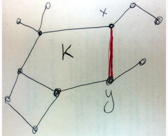
Then x and y are on the boundary of a common region in G_n and edge (x, y) splits K into two regions in G_{n+1}.

If G is a connected planar graph, then any plane graph depiction of G has $\boldsymbol{r} = \boldsymbol{e} - \boldsymbol{v} + 2$ regions.

Now, we assume that the theorem is true for any planar, connected graph of n-edges. i.e. if G_n is a planar, connected graph and $\boldsymbol{v}_n, \boldsymbol{e}_n$ and $\boldsymbol{r}_n$ number of vertices, edges and regions of G_n. Then

$$\boldsymbol{r}_n = \boldsymbol{e}_n - \boldsymbol{v}_n + 2.$$

Our goal now to show that we can make next step of induction and the theorem is true for any planar connected graph G_{n+1} of $n+1$ edges, i.e. $\boldsymbol{r}_{n+1} = \boldsymbol{e}_{n+1} - \boldsymbol{v}_{n+1} + 2$.
So let G_{n+1}, be planar connected graph of $n+1$ edges. Assume that we are drawing G_{n+1} edge by edge on the plane in such a way that on each step we have a connected graph. Let (x, y) be the last edge of G_{n+1} we draw (i.e. edge "number" $n+1$), the idea is to "remove" (x, y) and thus create a graph of just n edges and to use the assumption. Note that when we remove (x, y) we will have planar graph (subgraph of planar is planar) and connected (this is how we were drawing G_{n+1}). Still we will need to consider two cases (do we change regions or not):
In the first case x and y are were in G_n (the graph we draw before adding edge (x, y)):



Then x and y are on the boundary of a common region in G_n and edge (x, y) splits K into two regions in G_{n+1}. Then $\boldsymbol{v}_{n+1} = \boldsymbol{v}_n$,

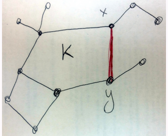
Theorem: Euler's Formula

If G is a connected planar graph, then any plane graph depiction of G has $\boldsymbol{r} = \boldsymbol{e} - \boldsymbol{v} + 2$ regions.

Now, we assume that the theorem is true for any planar, connected graph of n-edges. i.e. if G_n is a planar, connected graph and $\boldsymbol{v}_n, \boldsymbol{e}_n$ and $\boldsymbol{r}_n$ number of vertices, edges and regions of G_n. Then

$$\boldsymbol{r}_n = \boldsymbol{e}_n - \boldsymbol{v}_n + 2.$$

Our goal now to show that we can make next step of induction and the theorem is true for any planar connected graph G_{n+1} of $n+1$ edges, i.e. $\boldsymbol{r}_{n+1} = \boldsymbol{e}_{n+1} - \boldsymbol{v}_{n+1} + 2$.
So let G_{n+1}, be planar connected graph of $n+1$ edges. Assume that we are drawing G_{n+1} edge by edge on the plane in such a way that on each step we have a connected graph. Let (x, y) be the last edge of G_{n+1} we draw (i.e. edge "number" $n+1$), the idea is to "remove" (x, y) and thus create a graph of just n edges and to use the assumption. Note that when we remove (x, y) we will have planar graph (subgraph of planar is planar) and connected (this is how we were drawing G_{n+1}). Still we will need to consider two cases (do we change regions or not):
In the first case x and y are were in G_n (the graph we draw before adding edge (x, y)):



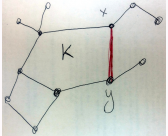
Then x and y are on the boundary of a common region in G_n and edge (x, y) splits K into two regions in G_{n+1}. Then $\boldsymbol{v}_{n+1} = \boldsymbol{v}_n$, $\boldsymbol{e}_{n+1} = \boldsymbol{e}_n + 1$

If G is a connected planar graph, then any plane graph depiction of G has $\boldsymbol{r} = \boldsymbol{e} - \boldsymbol{v} + 2$ regions.

Now, we assume that the theorem is true for any planar, connected graph of n-edges. i.e. if G_n is a planar, connected graph and $\boldsymbol{v}_n, \boldsymbol{e}_n$ and $\boldsymbol{r}_n$ number of vertices, edges and regions of G_n. Then

$$\boldsymbol{r}_n = \boldsymbol{e}_n - \boldsymbol{v}_n + 2.$$

Our goal now to show that we can make next step of induction and the theorem is true for any planar connected graph G_{n+1} of $n+1$ edges, i.e. $\boldsymbol{r}_{n+1} = \boldsymbol{e}_{n+1} - \boldsymbol{v}_{n+1} + 2$.
So let G_{n+1}, be planar connected graph of $n+1$ edges. Assume that we are drawing G_{n+1} edge by edge on the plane in such a way that on each step we have a connected graph. Let (x, y) be the last edge of G_{n+1} we draw (i.e. edge "number" $n+1$), the idea is to "remove" (x, y) and thus create a graph of just n edges and to use the assumption. Note that when we remove (x, y) we will have planar graph (subgraph of planar is planar) and connected (this is how we were drawing G_{n+1}). Still we will need to consider two cases (do we change regions or not):
In the first case x and y are were in G_n (the graph we draw before adding edge (x, y)):



Then x and y are on the boundary of a common region in G_n and edge (x, y) splits K into two regions in G_{n+1}. Then $\boldsymbol{v}_{n+1} = \boldsymbol{v}_n$, $\boldsymbol{e}_{n+1} = \boldsymbol{e}_n + 1$ and $\boldsymbol{r}_{n+1} = \boldsymbol{r}_n + 1$ now plug it into Euler's formula for G_n:

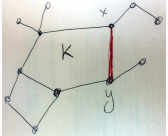
$$(\boldsymbol{r}_{n+1} - 1) = (\boldsymbol{e}_{n+1} - 1) - \boldsymbol{v}_{n+1} + 2.$$

If G is a connected planar graph, then any plane graph depiction of G has $\boldsymbol{r} = \boldsymbol{e} - \boldsymbol{v} + 2$ regions.

Now, we assume that the theorem is true for any planar, connected graph of n-edges. i.e. if G_n is a planar, connected graph and $\boldsymbol{v}_n, \boldsymbol{e}_n$ and $\boldsymbol{r}_n$ number of vertices, edges and regions of G_n. Then

$$\boldsymbol{r}_n = \boldsymbol{e}_n - \boldsymbol{v}_n + 2.$$

Our goal now to show that we can make next step of induction and the theorem is true for any planar connected graph G_{n+1} of $n+1$ edges, i.e. $\boldsymbol{r}_{n+1} = \boldsymbol{e}_{n+1} - \boldsymbol{v}_{n+1} + 2$.
So let G_{n+1}, be planar connected graph of $n+1$ edges. Assume that we are drawing G_{n+1} edge by edge on the plane in such a way that on each step we have a connected graph. Let (x, y) be the last edge of G_{n+1} we draw (i.e. edge "number" $n+1$), the idea is to "remove" (x, y) and thus create a graph of just n edges and to use the assumption. Note that when we remove (x, y) we will have planar graph (subgraph of planar is planar) and connected (this is how we were drawing G_{n+1}). Still we will need to consider two cases (do we change regions or not):
In the first case x and y are were in G_n (the graph we draw before adding edge (x, y)):



Then x and y are on the boundary of a common region in G_n and edge (x, y) splits K into two regions in G_{n+1}. Then $\boldsymbol{v}_{n+1} = \boldsymbol{v}_n$, $\boldsymbol{e}_{n+1} = \boldsymbol{e}_n + 1$ and $\boldsymbol{r}_{n+1} = \boldsymbol{r}_n + 1$ now plug it into Euler's formula for G_n:

$$(\boldsymbol{r}_{n+1} - 1) = (\boldsymbol{e}_{n+1} - 1) - \boldsymbol{v}_{n+1} + 2.$$

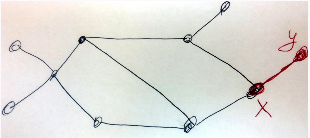
Cancelling -1 from both sides we get the formula we need for G_{n+1}. YES!!!

If G is a connected planar graph, then any plane graph depiction of G has $\boldsymbol{r} = \boldsymbol{e} - \boldsymbol{v} + 2$ regions.

Now, we assume that the theorem is true for any planar, connected graph of n-edges. i.e. if G_n is a planar, connected graph and $\boldsymbol{v}_n, \boldsymbol{e}_n$ and $\boldsymbol{r}_n$ number of vertices, edges and regions of G_n. Then

$$\boldsymbol{r}_n = \boldsymbol{e}_n - \boldsymbol{v}_n + 2.$$

Our goal now to show that we can make next step of induction and the theorem is true for any planar connected graph G_{n+1} of $n+1$ edges, i.e. $\boldsymbol{r}_{n+1} = \boldsymbol{e}_{n+1} - \boldsymbol{v}_{n+1} + 2$.
So let G_{n+1}, be planar connected graph of $n+1$ edges. Assume that we are drawing G_{n+1} edge by edge on the plane in such a way that on each step we have a connected graph. Let (x, y) be the last edge of G_{n+1} we draw (i.e. edge "number" $n+1$), the idea is to "remove" (x, y) and thus create a graph of just n edges and to use the assumption. Note that when we remove (x, y) we will have planar graph (subgraph of planar is planar) and connected (this is how we were drawing G_{n+1}). Still we will need to consider two cases (do we change regions or not):
In the second case one of x, y is not from G_n (it can be only one, the graphs are connected!):

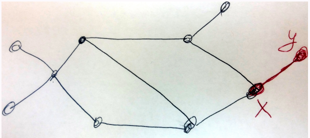


If G is a connected planar graph, then any plane graph depiction of G has $\boldsymbol{r} = \boldsymbol{e} - \boldsymbol{v} + 2$ regions.

Now, we assume that the theorem is true for any planar, connected graph of n-edges. i.e. if G_n is a planar, connected graph and $\boldsymbol{v}_n, \boldsymbol{e}_n$ and $\boldsymbol{r}_n$ number of vertices, edges and regions of G_n. Then

$$\boldsymbol{r}_n = \boldsymbol{e}_n - \boldsymbol{v}_n + 2.$$

Our goal now to show that we can make next step of induction and the theorem is true for any planar connected graph G_{n+1} of $n+1$ edges, i.e. $\boldsymbol{r}_{n+1} = \boldsymbol{e}_{n+1} - \boldsymbol{v}_{n+1} + 2$.
So let G_{n+1}, be planar connected graph of $n+1$ edges. Assume that we are drawing G_{n+1} edge by edge on the plane in such a way that on each step we have a connected graph. Let (x, y) be the last edge of G_{n+1} we draw (i.e. edge "number" $n+1$), the idea is to "remove" (x, y) and thus create a graph of just n edges and to use the assumption. Note that when we remove (x, y) we will have planar graph (subgraph of planar is planar) and connected (this is how we were drawing G_{n+1}). Still we will need to consider two cases (do we change regions or not):
In the second case one of x, y is not from G_n (it can be only one, the graphs are connected!):



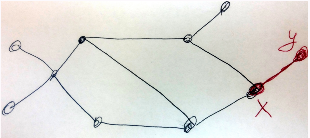
Then when we added (x, y) and created G_{n+1} we have NOT created new regions!

If G is a connected planar graph, then any plane graph depiction of G has $\boldsymbol{r} = \boldsymbol{e} - \boldsymbol{v} + 2$ regions.

Now, we assume that the theorem is true for any planar, connected graph of n-edges. i.e. if G_n is a planar, connected graph and $\boldsymbol{v}_n, \boldsymbol{e}_n$ and $\boldsymbol{r}_n$ number of vertices, edges and regions of G_n. Then

$$\boldsymbol{r}_n = \boldsymbol{e}_n - \boldsymbol{v}_n + 2.$$

Our goal now to show that we can make next step of induction and the theorem is true for any planar connected graph G_{n+1} of $n+1$ edges, i.e. $\boldsymbol{r}_{n+1} = \boldsymbol{e}_{n+1} - \boldsymbol{v}_{n+1} + 2$.
So let G_{n+1}, be planar connected graph of $n+1$ edges. Assume that we are drawing G_{n+1} edge by edge on the plane in such a way that on each step we have a connected graph. Let (x, y) be the last edge of G_{n+1} we draw (i.e. edge "number" $n+1$), the idea is to "remove" (x, y) and thus create a graph of just n edges and to use the assumption. Note that when we remove (x, y) we will have planar graph (subgraph of planar is planar) and connected (this is how we were drawing G_{n+1}). Still we will need to consider two cases (do we change regions or not):
In the second case one of x, y is not from G_n (it can be only one, the graphs are connected!):



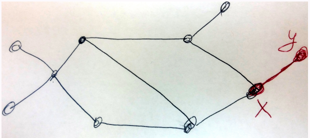
Then when we added (x, y) and created G_{n+1} we have NOT created new regions! Thus
$\boldsymbol{v}_{n+1} = \boldsymbol{v}_n + 1$,

If G is a connected planar graph, then any plane graph depiction of G has $\boldsymbol{r} = \boldsymbol{e} - \boldsymbol{v} + 2$ regions.

Now, we assume that the theorem is true for any planar, connected graph of n-edges. i.e. if G_n is a planar, connected graph and $\boldsymbol{v}_n, \boldsymbol{e}_n$ and $\boldsymbol{r}_n$ number of vertices, edges and regions of G_n. Then

$$\boldsymbol{r}_n = \boldsymbol{e}_n - \boldsymbol{v}_n + 2.$$

Our goal now to show that we can make next step of induction and the theorem is true for any planar connected graph G_{n+1} of $n+1$ edges, i.e. $\boldsymbol{r}_{n+1} = \boldsymbol{e}_{n+1} - \boldsymbol{v}_{n+1} + 2$.
So let G_{n+1}, be planar connected graph of $n+1$ edges. Assume that we are drawing G_{n+1} edge by edge on the plane in such a way that on each step we have a connected graph. Let (x, y) be the last edge of G_{n+1} we draw (i.e. edge "number" $n+1$), the idea is to "remove" (x, y) and thus create a graph of just n edges and to use the assumption. Note that when we remove (x, y) we will have planar graph (subgraph of planar is planar) and connected (this is how we were drawing G_{n+1}). Still we will need to consider two cases (do we change regions or not):
In the second case one of x, y is not from G_n (it can be only one, the graphs are connected!):



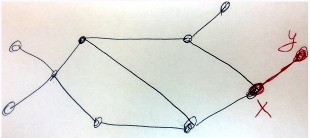
Then when we added (x, y) and created G_{n+1} we have NOT created new regions! Thus
$\boldsymbol{v}_{n+1} = \boldsymbol{v}_n + 1, \;\; \boldsymbol{e}_{n+1} = \boldsymbol{e}_n + 1$

If G is a connected planar graph, then any plane graph depiction of G has $\boldsymbol{r} = \boldsymbol{e} - \boldsymbol{v} + 2$ regions.

Now, we assume that the theorem is true for any planar, connected graph of n-edges. i.e. if G_n is a planar, connected graph and $\boldsymbol{v}_n, \boldsymbol{e}_n$ and $\boldsymbol{r}_n$ number of vertices, edges and regions of G_n. Then

$$\boldsymbol{r}_n = \boldsymbol{e}_n - \boldsymbol{v}_n + 2.$$

Our goal now to show that we can make next step of induction and the theorem is true for any planar connected graph G_{n+1} of $n+1$ edges, i.e. $\boldsymbol{r}_{n+1} = \boldsymbol{e}_{n+1} - \boldsymbol{v}_{n+1} + 2$.
So let G_{n+1}, be planar connected graph of $n+1$ edges. Assume that we are drawing G_{n+1} edge by edge on the plane in such a way that on each step we have a connected graph. Let (x, y) be the last edge of G_{n+1} we draw (i.e. edge "number" $n+1$), the idea is to "remove" (x, y) and thus create a graph of just n edges and to use the assumption. Note that when we remove (x, y) we will have planar graph (subgraph of planar is planar) and connected (this is how we were drawing G_{n+1}). Still we will need to consider two cases (do we change regions or not):
In the second case one of x, y is not from G_n (it can be only one, the graphs are connected!):



Then when we added (x, y) and created G_{n+1} we have NOT created new regions! Thus $\boldsymbol{v}_{n+1} = \boldsymbol{v}_n + 1$, $\boldsymbol{e}_{n+1} = \boldsymbol{e}_n + 1$ and $\boldsymbol{r}_{n+1} = \boldsymbol{r}_n$ now again plug it into Euler 'sformula for G_n:

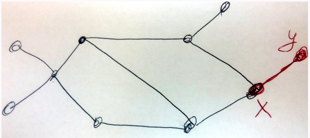
$$\boldsymbol{r}_{n+1} = (\boldsymbol{e}_{n+1} + 1) - (\boldsymbol{v}_{n+1} + 1) + 2.$$

If G is a connected planar graph, then any plane graph depiction of G has $\boldsymbol{r} = \boldsymbol{e} - \boldsymbol{v} + 2$ regions.

Now, we assume that the theorem is true for any planar, connected graph of n-edges. i.e. if G_n is a planar, connected graph and $\boldsymbol{v}_n, \boldsymbol{e}_n$ and $\boldsymbol{r}_n$ number of vertices, edges and regions of G_n. Then

$$\boldsymbol{r}_n = \boldsymbol{e}_n - \boldsymbol{v}_n + 2.$$

Our goal now to show that we can make next step of induction and the theorem is true for any planar connected graph G_{n+1} of $n+1$ edges, i.e. $\boldsymbol{r}_{n+1} = \boldsymbol{e}_{n+1} - \boldsymbol{v}_{n+1} + 2$.
So let G_{n+1}, be planar connected graph of $n+1$ edges. Assume that we are drawing G_{n+1} edge by edge on the plane in such a way that on each step we have a connected graph. Let (x, y) be the last edge of G_{n+1} we draw (i.e. edge "number" $n+1$), the idea is to "remove" (x, y) and thus create a graph of just n edges and to use the assumption. Note that when we remove (x, y) we will have planar graph (subgraph of planar is planar) and connected (this is how we were drawing G_{n+1}). Still we will need to consider two cases (do we change regions or not):
In the second case one of x, y is not from G_n (it can be only one, the graphs are connected!):



Then when we added (x, y) and created G_{n+1} we have NOT created new regions! Thus $\boldsymbol{v}_{n+1} = \boldsymbol{v}_n + 1$, $\boldsymbol{e}_{n+1} = \boldsymbol{e}_n + 1$ and $\boldsymbol{r}_{n+1} = \boldsymbol{r}_n$ now again plug it into Euler 'sformula for G_n:

$$\boldsymbol{r}_{n+1} = (\boldsymbol{e}_{n+1} + 1) - (\boldsymbol{v}_{n+1} + 1) + 2.$$

Cancelling $+1$ on the right we get the formula we need for G_{n+1}.

How many regions would there be in a plane graph with 10 vertices each of degree 3?

How many regions would there be in a plane graph with 10 vertices each of degree 3?

To use the Euler' formula we observe that $\boldsymbol{v} = 10$

How many regions would there be in a plane graph with 10 vertices each of degree 3?

To use the Euler' formula we observe that $\boldsymbol{v} = 10$ and $\boldsymbol{e} = 3 * 10/2 = 15$ thus

Euler's Formula – Example

How many regions would there be in a plane graph with 10 vertices each of degree 3?

To use the Euler' formula we observe that $\boldsymbol{v} = 10$ and $\boldsymbol{e} = 3 * 10/2 = 15$ thus

$$\boldsymbol{r} = \boldsymbol{e} - \boldsymbol{v} + 2 = 15 - 10 + 2 = 7.$$

Euler's Formula – Corollary

It would be great to use Euler's formula to check if a given graph is planar, indeed, IF the graph does not satisfy the formula it is not planar.

It would be great to use Euler's formula to check if a given graph is planar, indeed, IF the graph does not satisfy the formula it is not planar. The problem is "number of regions" –when we are given the graph and before we drawn it on the plane it may be impossible to find r, especially if it does not exists!

It would be great to use Euler's formula to check if a given graph is planar, indeed, IF the graph does not satisfy the formula it is not planar. The problem is "number of regions" –when we are given the graph and before we drawn it on the plane it may be impossible to find $\boldsymbol{r}$, especially if it does not exists! But here a super useful trick:

Corollary to Euler's formula:

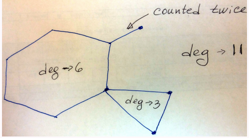
If G is a connected planar graph with $\boldsymbol{e} > 1$, then $\boldsymbol{e} \leq 3\boldsymbol{v} - 6$.

It would be great to use Euler's formula to check if a given graph is planar, indeed, IF the graph does not satisfy the formula it is not planar. The problem is "number of regions" –when we are given the graph and before we drawn it on the plane it may be impossible to find $\boldsymbol{r}$, especially if it does not exists! But here a super useful trick:

Corollary to Euler's formula:

If G is a connected planar graph with $\boldsymbol{e} > 1$, then $\boldsymbol{e} \leq 3\boldsymbol{v} - 6$.

Proof : Define the degree of a region to be the number of edges on its boundary (id edge occurs twice we count it twice).

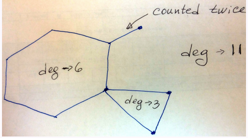


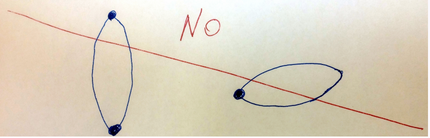
It would be great to use Euler's formula to check if a given graph is planar, indeed, IF the graph does not satisfy the formula it is not planar. The problem is "number of regions" –when we are given the graph and before we drawn it on the plane it may be impossible to find $\boldsymbol{r}$, especially if it does not exists! But here a super useful trick:

Corollary to Euler's formula:

If G is a connected planar graph with $\boldsymbol{e} > 1$, then $\boldsymbol{e} \leq 3\boldsymbol{v} - 6$.

Proof : Define the degree of a region to be the number of edges on its boundary (id edge occurs twice we count it twice).





Observe that each region on the plane must have degree greater or equal to 3 (we do not allow two edges between vertices and do not allow loops).

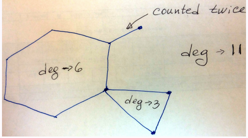
Euler's Formula – Corollary

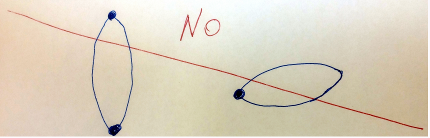
It would be great to use Euler's formula to check if a given graph is planar, indeed, IF the graph does not satisfy the formula it is not planar. The problem is "number of regions" –when we are given the graph and before we drawn it on the plane it may be impossible to find r, especially if it does not exists! But here a super useful trick:

Corollary to Euler's formula:

If G is a connected planar graph with $e > 1$, then $e \leq 3v - 6$.

Proof : Define the degree of a region to be the number of edges on its boundary (id edge occurs twice we count it twice).





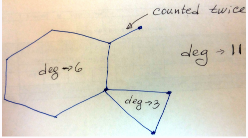
Observe that each region on the plane must have degree greater or equal to 3 (we do not allow two edges between vertices and do not allow loops). Thus the sum of degrees of all edges is at least $3r$.

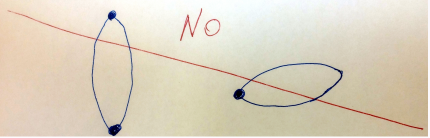
It would be great to use Euler's formula to check if a given graph is planar, indeed, IF the graph does not satisfy the formula it is not planar. The problem is "number of regions" –when we are given the graph and before we drawn it on the plane it may be impossible to find r, especially if it does not exists! But here a super useful trick:

Corollary to Euler's formula:

If G is a connected planar graph with $e > 1$, then $e \leq 3v - 6$.

Proof : Define the degree of a region to be the number of edges on its boundary (id edge occurs twice we count it twice).





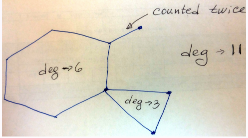
Observe that each region on the plane must have degree greater or equal to 3 (we do not allow two edges between vertices and do not allow loops). Thus the sum of degrees of all edges is at least $3r$. But we also note that every edge we counted twice, when were computing this sum, so it is $2e$,

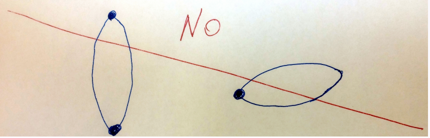
It would be great to use Euler's formula to check if a given graph is planar, indeed, IF the graph does not satisfy the formula it is not planar. The problem is "number of regions" –when we are given the graph and before we drawn it on the plane it may be impossible to find $\boldsymbol{r}$, especially if it does not exists! But here a super useful trick:

Corollary to Euler's formula:

If G is a connected planar graph with $\boldsymbol{e} > 1$, then $\boldsymbol{e} \leq 3\boldsymbol{v} - 6$.

Proof : Define the degree of a region to be the number of edges on its boundary (id edge occurs twice we count it twice).





Observe that each region on the plane must have degree greater or equal to 3 (we do not allow two edges between vertices and do not allow loops). Thus the sum of degrees of all edges is at least $3\boldsymbol{r}$. But we also note that every edge we counted twice, when were computing this sum, so it is $2\boldsymbol{e}$, thus

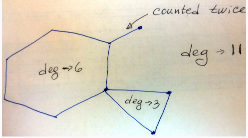
$$2\boldsymbol{e} \geq 3\boldsymbol{r} \text{ or } \frac{2}{3}\boldsymbol{e} \geq \boldsymbol{r},$$

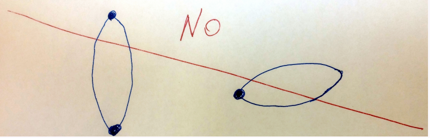
It would be great to use Euler's formula to check if a given graph is planar, indeed, IF the graph does not satisfy the formula it is not planar. The problem is "number of regions" –when we are given the graph and before we drawn it on the plane it may be impossible to find $\boldsymbol{r}$, especially if it does not exists! But here a super useful trick:

Corollary to Euler's formula:

If G is a connected planar graph with $\boldsymbol{e} > 1$, then $\boldsymbol{e} \leq 3\boldsymbol{v} - 6$.

Proof : Define the degree of a region to be the number of edges on its boundary (id edge occurs twice we count it twice).





Observe that each region on the plane must have degree greater or equal to 3 (we do not allow two edges between vertices and do not allow loops). Thus the sum of degrees of all edges is at least $3\boldsymbol{r}$. But we also note that every edge we counted twice, when were computing this sum, so it is $2\boldsymbol{e}$, thus

$$2\boldsymbol{e} \geq 3\boldsymbol{r} \text{ or } \frac{2}{3}\boldsymbol{e} \geq \boldsymbol{r},$$

Now we use Euler's formula to get

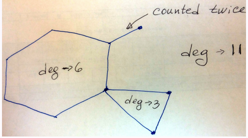
$$\frac{2}{3}\boldsymbol{e} \geq \boldsymbol{r} = \boldsymbol{e} - \boldsymbol{v} + 2 \text{ or } 0 \geq \frac{1}{3}\boldsymbol{e} - \boldsymbol{v} + 2,$$

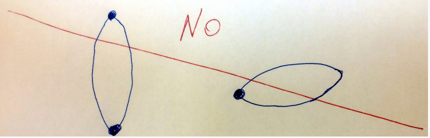
It would be great to use Euler's formula to check if a given graph is planar, indeed, IF the graph does not satisfy the formula it is not planar. The problem is "number of regions" –when we are given the graph and before we drawn it on the plane it may be impossible to find $\boldsymbol{r}$, especially if it does not exists! But here a super useful trick:

Corollary to Euler's formula:

If G is a connected planar graph with $\boldsymbol{e} > 1$, then $\boldsymbol{e} \leq 3\boldsymbol{v} - 6$.

Proof : Define the degree of a region to be the number of edges on its boundary (id edge occurs twice we count it twice).





Observe that each region on the plane must have degree greater or equal to 3 (we do not allow two edges between vertices and do not allow loops). Thus the sum of degrees of all edges is at least $3\boldsymbol{r}$. But we also note that every edge we counted twice, when were computing this sum, so it is $2\boldsymbol{e}$, thus

$$2\boldsymbol{e} \geq 3\boldsymbol{r} \text{ or } \frac{2}{3}\boldsymbol{e} \geq \boldsymbol{r},$$

Now we use Euler's formula to get

$$\frac{2}{3}\boldsymbol{e} \geq \boldsymbol{r} = \boldsymbol{e} - \boldsymbol{v} + 2 \text{ or } 0 \geq \frac{1}{3}\boldsymbol{e} - \boldsymbol{v} + 2,$$

which is exactly $\boldsymbol{e} \leq 3\boldsymbol{v} - 6$.

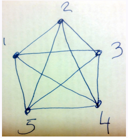
Corollary to Euler's formula:

If G is a connected planar graph with $\boldsymbol{e} > 1$, then $\boldsymbol{e} \leq 3\boldsymbol{v} - 6$.

Corollary to Euler's formula:

If G is a connected planar graph with $\boldsymbol{e} > 1$, then $\boldsymbol{e} \leq 3\boldsymbol{v} - 6$.

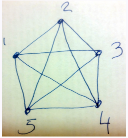
So lets apply it to our old friend K_5:



Corollary to Euler's formula:

If G is a connected planar graph with $e > 1$, then $e \leq 3v - 6$.

So lets apply it to our old friend K_5:

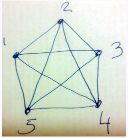


Here $v = 5$ and $e = 5 * 4/2 = 10$

Corollary to Euler's formula:

If G is a connected planar graph with $\boldsymbol{e} > 1$, then $\boldsymbol{e} \leq 3\boldsymbol{v} - 6$.

So lets apply it to our old friend K_5:

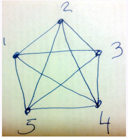


Here $\boldsymbol{v} = 5$ and $\boldsymbol{e} = 5 * 4/2 = 10$ and $3\boldsymbol{v} - 6 = 9$ which is less then 10

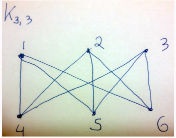
Corollary to Euler's formula:

If G is a connected planar graph with $\boldsymbol{e} > 1$, then $\boldsymbol{e} \leq 3\boldsymbol{v} - 6$.

So lets apply it to our old friend K_5:



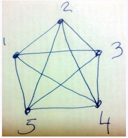
Here $\boldsymbol{v} = 5$ and $\boldsymbol{e} = 5 * 4/2 = 10$ and $3\boldsymbol{v} - 6 = 9$ which is less then 10 so K_5 CAN NOT BE PLANAR!
Lets apply it to our other old friend $K_{3,3}$:



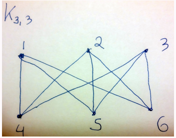
Corollary to Euler's formula:

If G is a connected planar graph with $\boldsymbol{e} > 1$, then $\boldsymbol{e} \leq 3\boldsymbol{v} - 6$.

So lets apply it to our old friend K_5:



Here $\boldsymbol{v} = 5$ and $\boldsymbol{e} = 5 * 4/2 = 10$ and $3\boldsymbol{v} - 6 = 9$ which is less then 10 so K_5 CAN NOT BE PLANAR!
Lets apply it to our other old friend $K_{3,3}$:

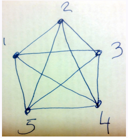


Here $\boldsymbol{v} = 6$ and $\boldsymbol{e} = 6 * 3/2 = 9$

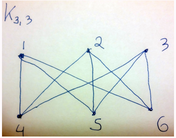
Corollary to Euler's formula:

If G is a connected planar graph with $\boldsymbol{e} > 1$, then $\boldsymbol{e} \leq 3\boldsymbol{v} - 6$.

So lets apply it to our old friend K_5:



Here $\boldsymbol{v} = 5$ and $\boldsymbol{e} = 5 * 4/2 = 10$ and $3\boldsymbol{v} - 6 = 9$ which is less then 10 so K_5 CAN NOT BE PLANAR!
Lets apply it to our other old friend $K_{3,3}$:

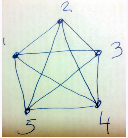


Here $\boldsymbol{v} = 6$ and $\boldsymbol{e} = 6 * 3/2 = 9$ and $3\boldsymbol{v} - 6 = 12$

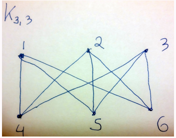
Corollary to Euler's formula:

If G is a connected planar graph with $\boldsymbol{e} > 1$, then $\boldsymbol{e} \leq 3\boldsymbol{v} - 6$.

So lets apply it to our old friend K_5:



Here $\boldsymbol{v} = 5$ and $\boldsymbol{e} = 5 * 4/2 = 10$ and $3\boldsymbol{v} - 6 = 9$ which is less then 10 so K_5 CAN NOT BE PLANAR!
Lets apply it to our other old friend $K_{3,3}$:



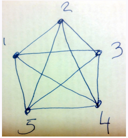
Here $\boldsymbol{v} = 6$ and $\boldsymbol{e} = 6 * 3/2 = 9$ and $3\boldsymbol{v} - 6 = 12$ and .... it works $12 > 9$.

Euler's Formula – Corollary – Example

Corollary to Euler's formula:

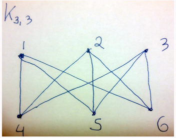
If G is a connected planar graph with $e > 1$, then $e \leq 3v - 6$.

So lets apply it to our old friend K_5:



Here $v = 5$ and $e = 5 * 4/2 = 10$ and $3v - 6 = 9$ which is less then 10 so K_5 CAN NOT BE PLANAR!

Lets apply it to our other old friend $K_{3,3}$:

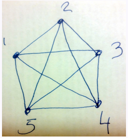


Here $v = 6$ and $e = 6 * 3/2 = 9$ and $3v - 6 = 12$ and .... it works $12 > 9$. SO DOES IT TELL US THAT $K_{3,3}$ is PLANAR???

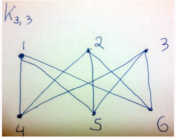
Corollary to Euler's formula:

If G is a connected planar graph with $\boldsymbol{e} > 1$, then $\boldsymbol{e} \leq 3\boldsymbol{v} - 6$.

So lets apply it to our old friend K_5:



Here $\boldsymbol{v} = 5$ and $\boldsymbol{e} = 5 * 4/2 = 10$ and $3\boldsymbol{v} - 6 = 9$ which is less then 10 so K_5 CAN NOT BE PLANAR!
Lets apply it to our other old friend $K_{3,3}$:

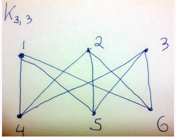


Here $\boldsymbol{v} = 6$ and $\boldsymbol{e} = 6 * 3/2 = 9$ and $3\boldsymbol{v} - 6 = 12$ and .... it works $12 > 9$. SO DOES IT TELL US THAT $K_{3,3}$ is PLANAR??? NO NO and NO **(the corollary does NOT let us unidentify planar graphs!! it only helps to catch non-planar!)**

Corollary to Euler's formula:

If G is a connected planar graph with $e > 1$, then $e \leq 3v - 6$.

Lets apply it to our other old friend $K_{3,3}$:

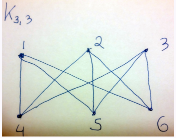


Here $v = 6$ and $e = 6 * 3/2 = 9$ and $3v - 6 = 12$ and .... it works $10 > 9$. SO DOES IT TELL US THAT $K_{3,3}$ is PLANAR??? NO NO and NO **(the corollary does NOT let us unidentify planar graphs!! it only helps to catch non-planar!)**

Corollary to Euler's formula:

If G is a connected planar graph with $\boldsymbol{e} > 1$, then $\boldsymbol{e} \leq 3\boldsymbol{v} - 6$.

Lets apply it to our other old friend $K_{3,3}$:

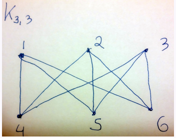


Here $\boldsymbol{v} = 6$ and $\boldsymbol{e} = 6 * 3/2 = 9$ and $3\boldsymbol{v} - 6 = 12$ and .... it works $10 > 9$. SO DOES IT TELL US THAT $K_{3,3}$ is PLANAR??? NO NO and NO **(the corollary does NOT let us unidentify planar graphs!! it only helps to catch non-planar!)**
Just to show that Euler's formula is very strong let us see how we can still use it to solve $K_{3,3}$ problem.

Corollary to Euler's formula:

If G is a connected planar graph with $\boldsymbol{e} > 1$, then $\boldsymbol{e} \leq 3\boldsymbol{v} - 6$.

Lets apply it to our other old friend $K_{3,3}$:

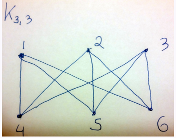


Here $\boldsymbol{v} = 6$ and $\boldsymbol{e} = 6 * 3/2 = 9$ and $3\boldsymbol{v} - 6 = 12$ and .... it works $10 > 9$. SO DOES IT TELL US THAT $K_{3,3}$ is PLANAR??? NO NO and NO **(the corollary does NOT let us unidentify planar graphs!! it only helps to catch non-planar!)**
Just to show that Euler's formula is very strong let us see how we can still use it to solve $K_{3,3}$ problem. Notice that $K_{3,3}$ is a bipartite graph.

Corollary to Euler's formula:

If G is a connected planar graph with $\boldsymbol{e} > 1$, then $\boldsymbol{e} \leq 3\boldsymbol{v} - 6$.

Lets apply it to our other old friend $K_{3,3}$:

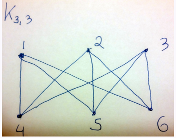


Here $\boldsymbol{v} = 6$ and $\boldsymbol{e} = 6 * 3/2 = 9$ and $3\boldsymbol{v} - 6 = 12$ and .... it works $10 > 9$. SO DOES IT TELL US THAT $K_{3,3}$ is PLANAR??? NO NO and NO **(the corollary does NOT let us unidentify planar graphs!! it only helps to catch non-planar!)**
Just to show that Euler's formula is very strong let us see how we can still use it to solve $K_{3,3}$ problem. Notice that $K_{3,3}$ is a bipartite graph. We proved that those guys may have circuits of even length only!

Corollary to Euler's formula:

If G is a connected planar graph with $e > 1$, then $e \leq 3v - 6$.

Lets apply it to our other old friend $K_{3,3}$:



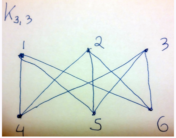
Here $v = 6$ and $e = 6 * 3/2 = 9$ and $3v - 6 = 12$ and .... it works $10 > 9$. SO DOES IT TELL US THAT $K_{3,3}$ is PLANAR??? NO NO and NO **(the corollary does NOT let us unidentify planar graphs!! it only helps to catch non-planar!)**
Just to show that Euler's formula is very strong let us see how we can still use it to solve $K_{3,3}$ problem. Notice that $K_{3,3}$ is a bipartite graph. We proved that those guys may have circuits of even length only! Thus a shortest circuit we may have is of length 4 (and NOT 3 as for a general graph!).

Euler's Formula – Corollary – Example

Corollary to Euler's formula:

If G is a connected planar graph with $\boldsymbol{e} > 1$, then $\boldsymbol{e} \leq 3\boldsymbol{v} - 6$.

Lets apply it to our other old friend $K_{3,3}$:

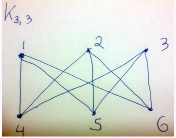


Here $\boldsymbol{v} = 6$ and $\boldsymbol{e} = 6 * 3/2 = 9$ and $3\boldsymbol{v} - 6 = 12$ and .... it works $10 > 9$. SO DOES IT TELL US THAT $K_{3,3}$ is PLANAR??? NO NO and NO **(the corollary does NOT let us unidentify planar graphs!! it only helps to catch non-planar!)**
Just to show that Euler's formula is very strong let us see how we can still use it to solve $K_{3,3}$ problem. Notice that $K_{3,3}$ is a bipartite graph. We proved that those guys may have circuits of even length only! Thus a shortest circuit we may have is of length 4 (and NOT 3 as for a general graph!). In the proof of the corollary we computed the sum of all degrees of regions (and the estimate we got was greater then $3\boldsymbol{r}$),

Corollary to Euler's formula:

If G is a connected planar graph with $\boldsymbol{e} > 1$, then $\boldsymbol{e} \leq 3\boldsymbol{v} - 6$.

Lets apply it to our other old friend $K_{3,3}$:

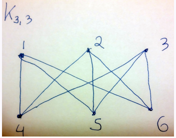


Here $\boldsymbol{v} = 6$ and $\boldsymbol{e} = 6 * 3/2 = 9$ and $3\boldsymbol{v} - 6 = 12$ and .... it works $10 > 9$. SO DOES IT TELL US THAT $K_{3,3}$ is PLANAR??? NO NO and NO **(the corollary does NOT let us unidentify planar graphs!! it only helps to catch non-planar!)**
Just to show that Euler's formula is very strong let us see how we can still use it to solve $K_{3,3}$ problem. Notice that $K_{3,3}$ is a bipartite graph. We proved that those guys may have circuits of even length only! Thus a shortest circuit we may have is of length 4 (and NOT 3 as for a general graph!). In the proof of the corollary we computed the sum of all degrees of regions (and the estimate we got was greater then $3\boldsymbol{r}$), now we can improve this estimate for a bipartite graph – $4\boldsymbol{r}$

Corollary to Euler's formula:

If G is a connected planar graph with $e > 1$, then $e \leq 3v - 6$.

Lets apply it to our other old friend $K_{3,3}$:

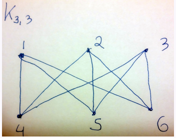


Here $v = 6$ and $e = 6 * 3/2 = 9$ and $3v - 6 = 12$ and .... it works $10 > 9$. SO DOES IT TELL US THAT $K_{3,3}$ is PLANAR??? NO NO and NO **(the corollary does NOT let us unidentify planar graphs!! it only helps to catch non-planar!)**
Just to show that Euler's formula is very strong let us see how we can still use it to solve $K_{3,3}$ problem. Notice that $K_{3,3}$ is a bipartite graph. We proved that those guys may have circuits of even length only! Thus a shortest circuit we may have is of length 4 (and NOT 3 as for a general graph!). In the proof of the corollary we computed the sum of all degrees of regions (and the estimate we got was greater then $3r$), now we can improve this estimate for a bipartite graph – $4r$ and get in this special case inequality $2e \geq 4r$ or $\frac{1}{2}e \geq r$.

Corollary to Euler's formula:

If G is a connected planar graph with $\boldsymbol{e} > 1$, then $\boldsymbol{e} \leq 3\boldsymbol{v} - 6$.

Lets apply it to our other old friend $K_{3,3}$:

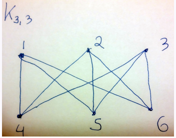


Here $\boldsymbol{v} = 6$ and $\boldsymbol{e} = 6 * 3/2 = 9$ and $3\boldsymbol{v} - 6 = 12$ and .... it works $10 > 9$. SO DOES IT TELL US THAT $K_{3,3}$ is PLANAR??? NO NO and NO **(the corollary does NOT let us unidentify planar graphs!! it only helps to catch non-planar!)**
Just to show that Euler's formula is very strong let us see how we can still use it to solve $K_{3,3}$ problem. Notice that $K_{3,3}$ is a bipartite graph. We proved that those guys may have circuits of even length only! Thus a shortest circuit we may have is of length 4 (and NOT 3 as for a general graph!). In the proof of the corollary we computed the sum of all degrees of regions (and the estimate we got was greater then $3\boldsymbol{r}$), now we can improve this estimate for a bipartite graph – $4\boldsymbol{r}$ and get in this special case inequality $2\boldsymbol{e} \geq 4\boldsymbol{r}$ or $\frac{1}{2}\boldsymbol{e} \geq \boldsymbol{r}$. Plug it in Euler formula
$\frac{1}{2}\boldsymbol{e} \geq \boldsymbol{r} = \boldsymbol{e} - \boldsymbol{v} + 2$

Corollary to Euler's formula:

If G is a connected planar graph with $\boldsymbol{e} > 1$, then $\boldsymbol{e} \leq 3\boldsymbol{v} - 6$.

Lets apply it to our other old friend $K_{3,3}$:



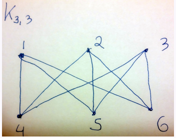
Here $\boldsymbol{v} = 6$ and $\boldsymbol{e} = 6 * 3/2 = 9$ and $3\boldsymbol{v} - 6 = 12$ and .... it works $10 > 9$. SO DOES IT TELL US THAT $K_{3,3}$ is PLANAR??? NO NO and NO **(the corollary does NOT let us unidentify planar graphs!! it only helps to catch non-planar!)**
Just to show that Euler's formula is very strong let us see how we can still use it to solve $K_{3,3}$ problem. Notice that $K_{3,3}$ is a bipartite graph. We proved that those guys may have circuits of even length only! Thus a shortest circuit we may have is of length 4 (and NOT 3 as for a general graph!). In the proof of the corollary we computed the sum of all degrees of regions (and the estimate we got was greater then $3\boldsymbol{r}$), now we can improve this estimate for a bipartite graph – $4\boldsymbol{r}$ and get in this special case inequality $2\boldsymbol{e} \geq 4\boldsymbol{r}$ or $\frac{1}{2}\boldsymbol{e} \geq \boldsymbol{r}$. Plug it in Euler formula $\frac{1}{2}\boldsymbol{e} \geq \boldsymbol{r} = \boldsymbol{e} - \boldsymbol{v} + 2$ and get a **special case of the corollary for bipartite connected planar graphs**:

$$\boldsymbol{e} \leq 2\boldsymbol{v} - 4$$

Corollary to Euler's formula:

If G is a connected planar graph with $\boldsymbol{e} > 1$, then $\boldsymbol{e} \leq 3\boldsymbol{v} - 6$.

Lets apply it to our other old friend $K_{3,3}$:



Here $\boldsymbol{v} = 6$ and $\boldsymbol{e} = 6 * 3/2 = 9$ and $3\boldsymbol{v} - 6 = 12$ and .... it works $10 > 9$. SO DOES IT TELL US THAT $K_{3,3}$ is PLANAR??? NO NO and NO **(the corollary does NOT let us unidentify planar graphs!! it only helps to catch non-planar!)**
Just to show that Euler's formula is very strong let us see how we can still use it to solve $K_{3,3}$ problem. Notice that $K_{3,3}$ is a bipartite graph. We proved that those guys may have circuits of even length only! Thus a shortest circuit we may have is of length 4 (and NOT 3 as for a general graph!). In the proof of the corollary we computed the sum of all degrees of regions (and the estimate we got was greater then $3\boldsymbol{r}$), now we can improve this estimate for a bipartite graph – $4\boldsymbol{r}$ and get in this special case inequality $2\boldsymbol{e} \geq 4\boldsymbol{r}$ or $\frac{1}{2}\boldsymbol{e} \geq \boldsymbol{r}$. Plug it in Euler formula $\frac{1}{2}\boldsymbol{e} \geq \boldsymbol{r} = \boldsymbol{e} - \boldsymbol{v} + 2$ and get a **special case of the corollary for bipartite connected planar graphs**:

$$\boldsymbol{e} \leq 2\boldsymbol{v} - 4$$

Now it is easy to check that $K_{3,3}$ does not satisfy it (indeed $9 \not\leq 2 * 6 - 4$) and thus it is not planar.